Rainbow Connection in Oriented Graphs

An Overview of Dorbec et al. 2014

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Math 522

What is the “rainbow connection”?

All graphs in this paper are oriented and strong.

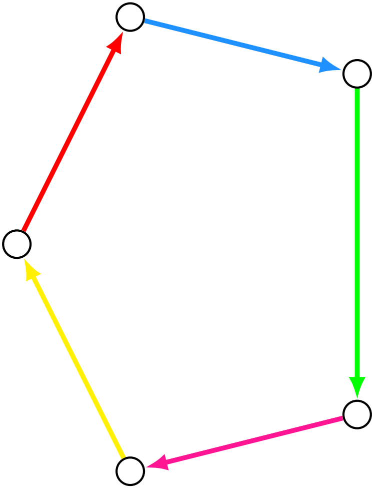
What is the “rainbow connection”?

All graphs in this paper are oriented and strong. Recall that a graph is *strong* if there exists a directed path between any two vertices.

What is the “rainbow connection”?

The rainbow connection number of a strong graph G, denoted $\vec{rc}(G)$, is the minimum edge-coloring of G such that there exists a path P between any two vertices, where every edge in P is a different color.

What is the “rainbow connection”?



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1) Which graphs G have $\vec{rc}(G) = n$?

2) What is the rainbow connection number of tournaments?

Which graphs G have $\vec{rc}(G) = n$?

How can we characterize $\vec{rc}(G)$?

Which graphs G have $\vec{rc}(G) = n$?

Some preliminary observations

Theorem

For any strong graph G, $\vec{rc}(G) \geq diam(G)$.

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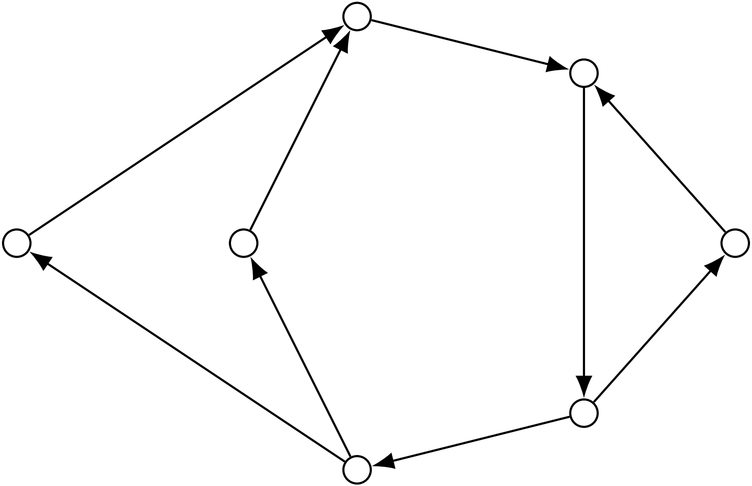
Theorem

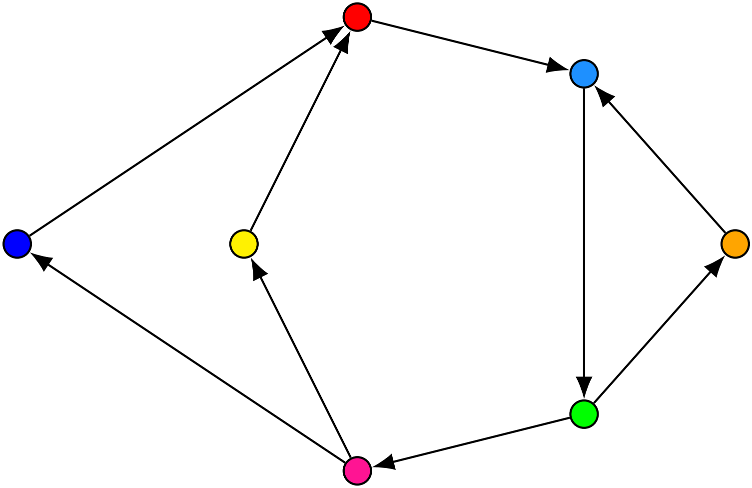
For any strong graph G, $\vec{rc}(G) \leq n(G)$.

Which graphs G have $\vec{rc}(G) = n$?



Which graphs G have $\vec{rc}(G) = n$?

(1) Color each vertex differently...



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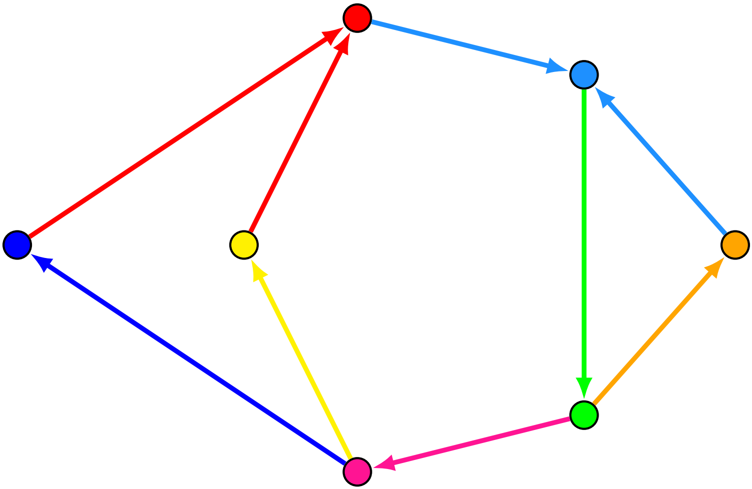
(2) ...and color all edges uv the color of v.



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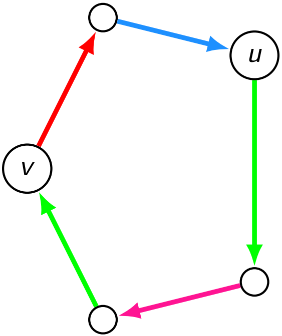
So, we can rainbow-edge-color any strong graph with at most n colors. Can we do better?

Which graphs G have $\vec{rc}(G) = n$?

Not if G is a cycle...



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Theorem

Let G be a strong oriented graph on n vertices, with arcs $x'x$ and $y'y$, where $x \neq y$ and x and y have in-degree 1. Then, if $x'x$, $y'y$ have the "path property", G has rainbow coloring number at most $n-1$ (i.e. $\vec{rc}(G) \leq n-1$).

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All *minimally strongly connected* graphs have such arcs $x'x$ and $y'y$.

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Theorem

All *minimally strongly connected* graphs have such arcs $x'x$ and $y'y$.

Theorem

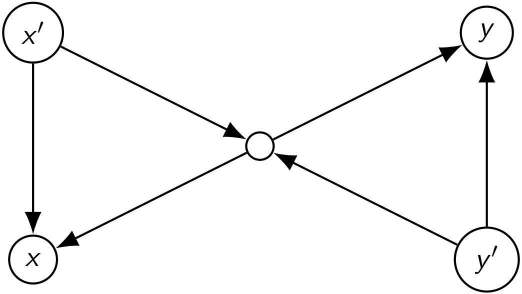
If G is a *minimally strongly connected* (MSC) oriented graph on n vertices, and G is *not* a cycle, then G has rainbow connection number at most $n - 1$ (i.e. $\vec{rc}(G) \leq n - 1$).

Which graphs G have $\vec{rc}(G) = n$?

Introducing: The Path Property

Definition

Two arcs $x'x$ and $y'y$ in G have the *path property* if there exists a path from x' to y that does not include $x'x$ *and* a path from y' to x that does not include $y'y$.



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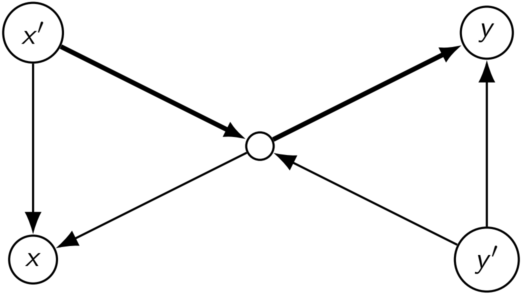

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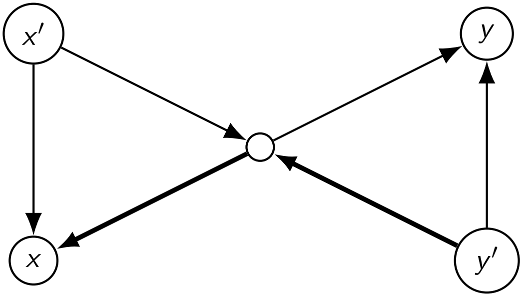

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A coloring scheme for MSC graphs

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A coloring scheme for MSC graphs

1) Find two edges $x'x$ and $y'y$ that satisfy the "path property" and the in-degree condition.

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A coloring scheme for MSC graphs

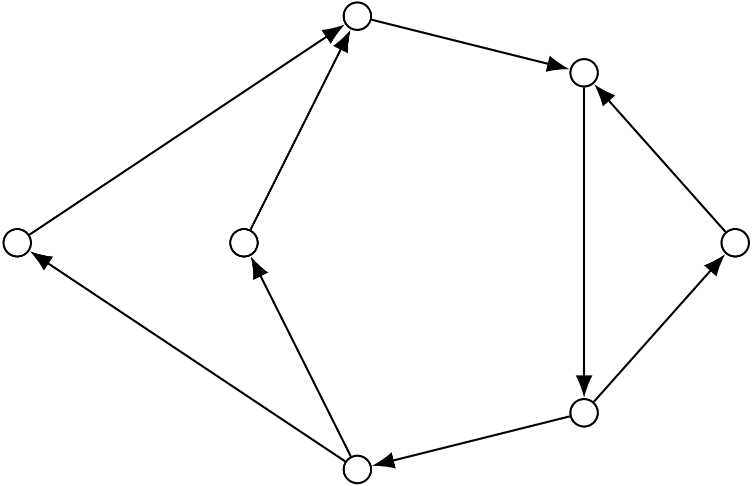
1) Find two edges $x'x$ and $y'y$ that satisfy the "path property" and the in-degree condition.

2) Color x and y with color 1, and each other vertex with a unique color in $\{2, ..., n-1\}$.

3) Color all edges going into a vertex the color of that vertex.

Which graphs G have $\vec{rc}(G) = n$?

An example



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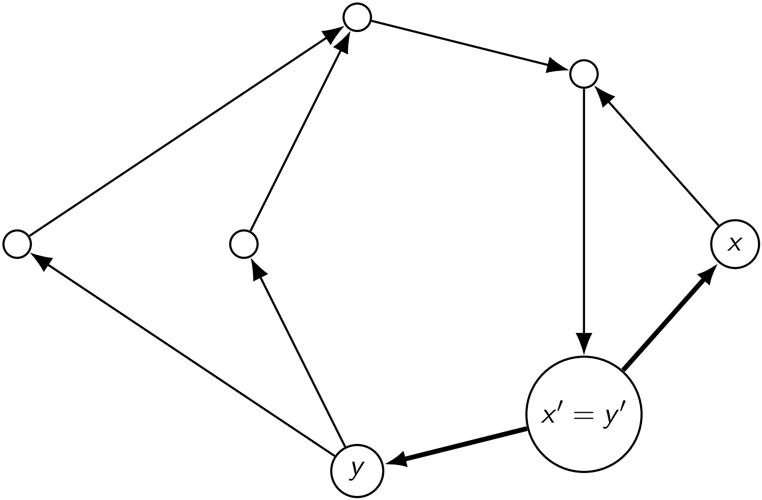

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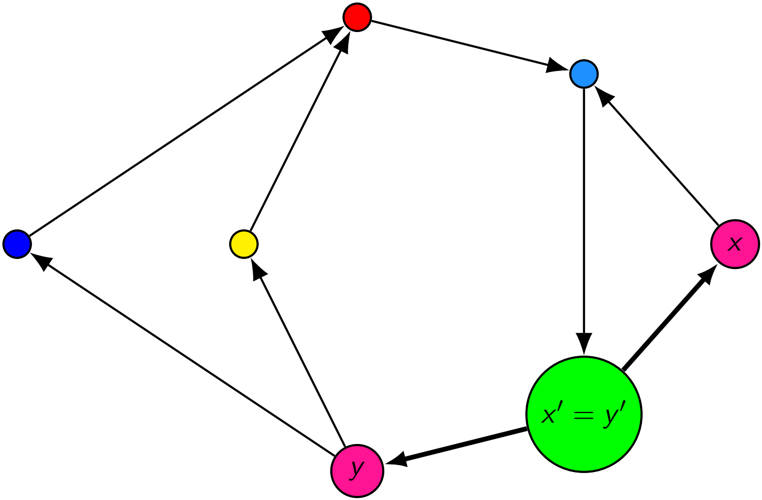

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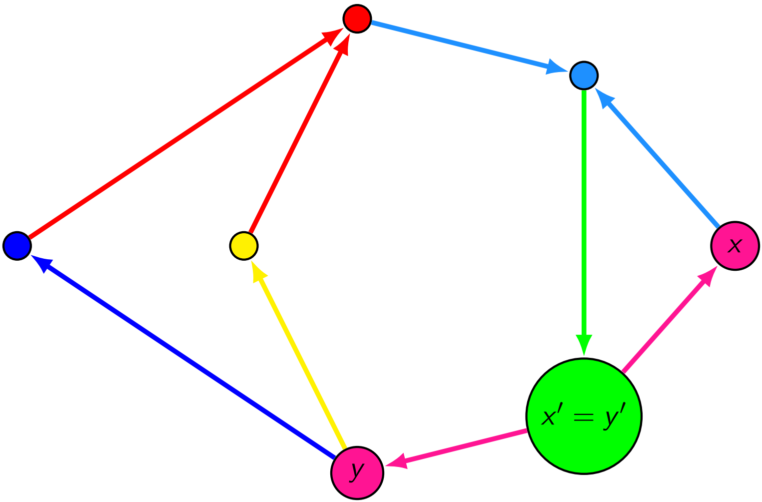

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A slightly easier characterization

Note that:

Theorem

For any spanning subgraph H of G, $\vec{rc}(H) \geq \vec{rc}(G)$.

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For any spanning subgraph H of G, $\vec{rc}(H) \geq \vec{rc}(G)$.

Also, note that if G does not have any Hamiltonian cycles, then either G is MSC and not a cycle, or some spanning subgraph H of G is MSC and not a cycle. In either case, this implies that $\vec{rc}(G) \leq n - 1$.

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Also, note that if G does not have any Hamiltonian cycles, then either G is MSC and not a cycle, or some spanning subgraph H of G is MSC and not a cycle. In either case, this implies that $\vec{rc}(G) \leq n - 1$.

Theorem

If G is not Hamiltonian, then $\vec{rc}(G) \leq n - 1$.

Which graphs G have $\vec{rc}(G) = n$?

A slightly easier characterization

So, which graphs actually have $\vec{rc}(G) = n$?

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A slightly easier characterization

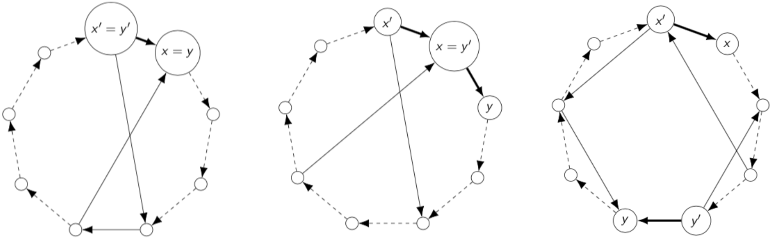
So, which graphs actually have $\vec{rc}(G) = n$?

Theorem

A graph G has $\vec{rc}(G) = n$ iff G is Hamiltonian and no cycle contains arcs that satisfy the path property.

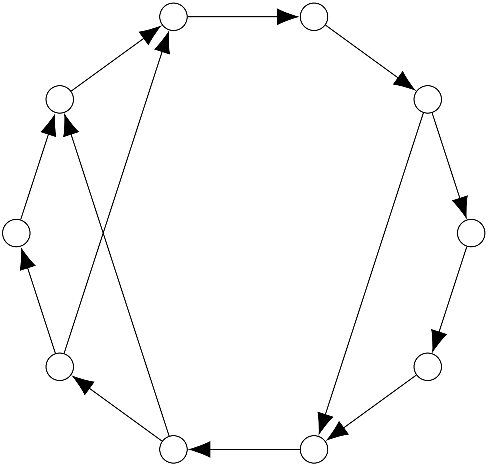
Which graphs G have $\vec{rc}(G) = n$?

Unallowable subgraphs



Which graphs G have $\vec{rc}(G) = n$?

An actual example



What is the rainbow connection number of tournaments?

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(Again, we will assume that T is strong.)

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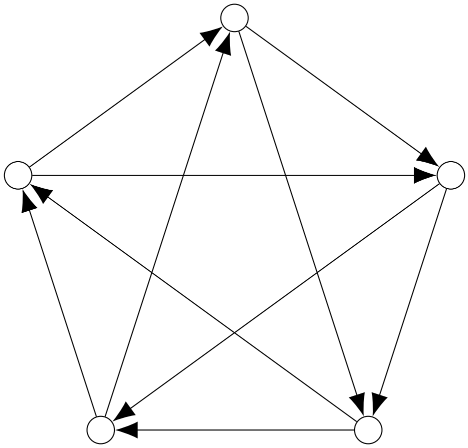
A proof by example



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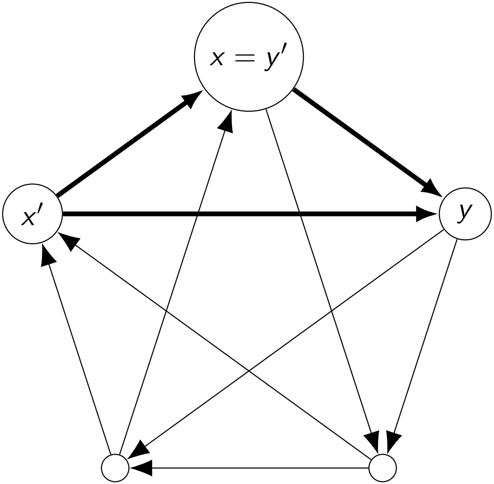

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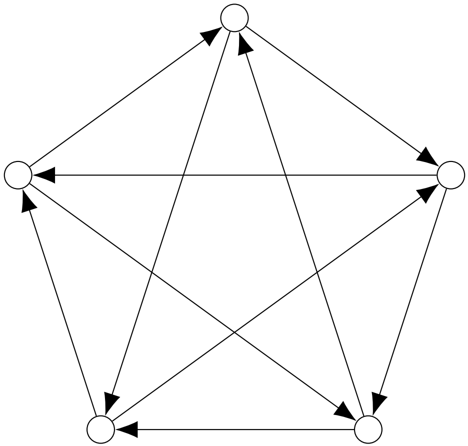

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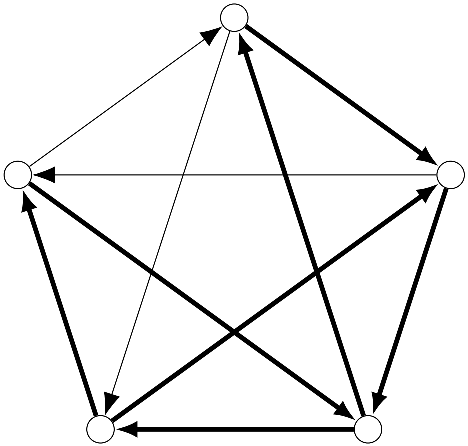

What is the rainbow connection number of tournaments?

One final theorem

Theorem

For any strong tournament T, $diam(T) \leq \vec{rc}(T) \leq diam(T) + 2$.

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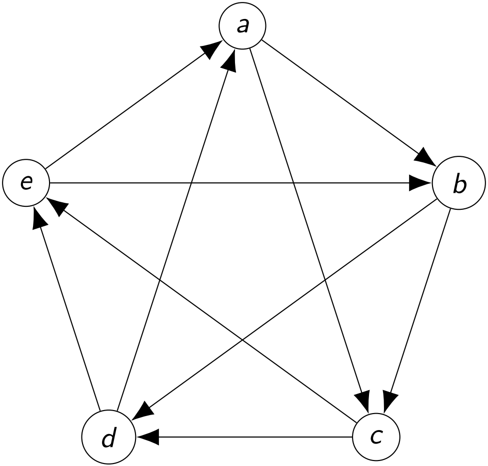
One final proof by example



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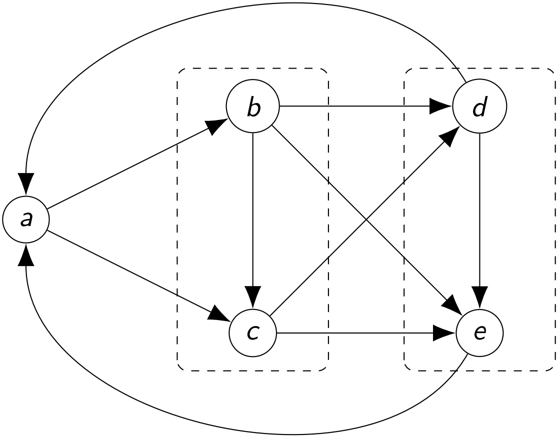

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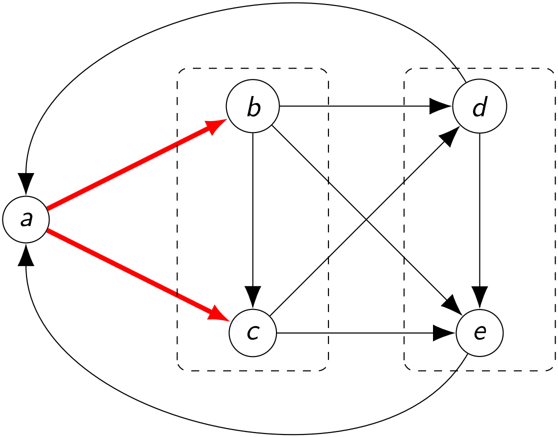

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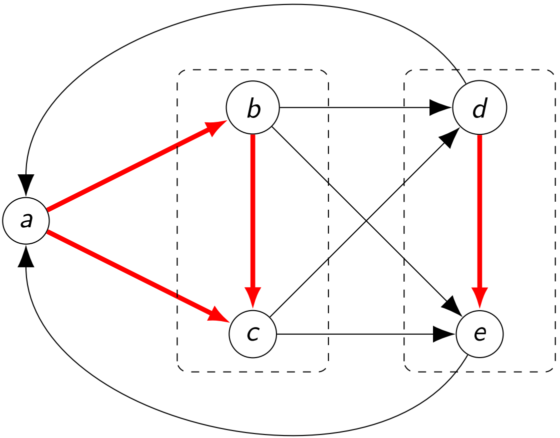

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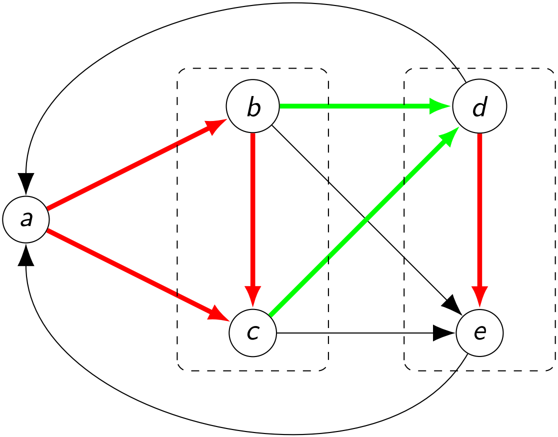

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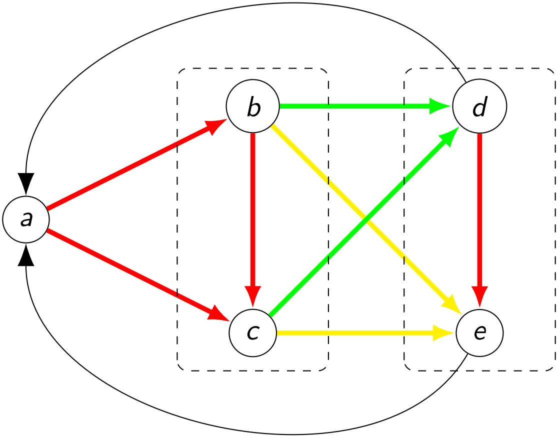

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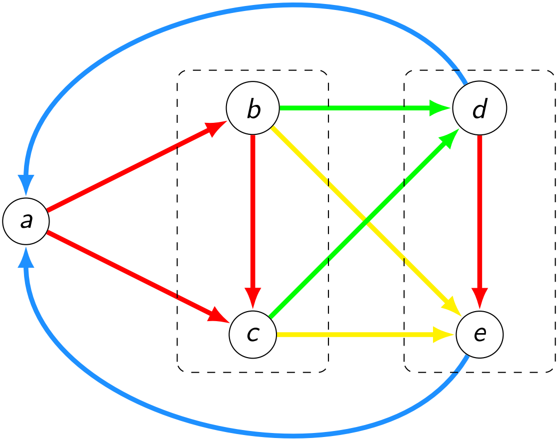

Open Questions

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2) Which tournaments have rainbow connection number 2?

Thanks!

Dorbec, P., Schiermeyer, I., Sidorowicz, E., and Sopena, E. (2014). *Rainbow connection in oriented graphs*. Discrete Applied Mathematics, **179**, 69-78.