Regular and Semi-Regular Tilings

Adapted from A. Gardiner's *Mathematical Puzzling* by Abel Romer

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1 Organization

The goal of this lesson is gain a foundational understanding of regular and semi-regular tilings on the plane and in 3D space, and use this understanding to begin to explore basic concepts in group theory. This exploration will center around five basic questions:

1. How many regular tilings of the plane are there?
2. How many regular polyhedra are there?
3. How many semi-regular tilings of the plane are there?
4. How many semi-regular polyhedra are there?
5. How many regular and semi-regular tiling of space are there?

Each section will provide guiding questions for exploring each of these questions, as well as the answers and potential strategies for reaching them. It will also include possible extensions and alternate directions of study.

2 Lesson

2.1 Task

Consider a regular polygon. Imagine labelling the vertices. How many unique orientations can you find? What types of symmetries did you use?

A regular tessellation is an arrangement of one type of regular polygon on the plane, such that all edges touch without gaps or overlaps and each vertex looks the same. How many regular tessellations exist? Can you prove it?

Consider the regular tessellations that you found. What type of symmetries do they have? Are they the same as their constituent polygons? Are there more?

2.2 Explanation

The first question aims to begin identifying characteristics of symmetry groups. The symmetry groups of the regular polygons up to size n are the dihedral groups $D_3, D_4, ..., D_n$. There are $2n$ unique symmetries of a n-sided polygon. The two symmetries used are rotational and reflective.

There are three regular tessellations. This can be shown by recognizing that the total sum of angles at each vertex must equal 360 degrees. Therefore, only regular polygons whose vertex angles divide 360 are candidates. This leaves us with the triangle, square and hexagon.

The triangle and hexagon tessellations have rotational, reflective and translational symmetries. They fall into group $p6m$ of the wallpaper groups. The square tessellation has the same symmetry types. It differs in that it has rotational symmetries of order 2 and 4, whereas the other tessellations have rotational symmetries of order 2, 3, and 6. It falls into group $p4m$.

2.3 Task

A regular polyhedron is a 3D solid that has sides made from identical regular polygons, and the same vertex pattern for each vertex. How many regular polyhedra can you find? Can you prove that this is all of them?

What type of symmetries can you find in the regular polyhedra? What similarities do you notice between polyhedra? How many unique orientations are there of the tetrahedron? The cube? The octahedron?

2.4 Explanation

There are five regular polyhedra. The key realization to make here is that unlike the tessellations, the sum of angles at any vertex must be strictly less than 360 degrees. Additionally, there must be at least three faces meeting at any vertex, or otherwise the faces would have to curve or extend infinitely in one direction. Given these constraints, the only possible polyhedra are formed from a triangle, a square and a pentagon.

There are rotational symmetries along the three axes of rotation. The cube and octahedron, and dodecahedron and icosahedron are duals. Dual platonic solids have the same number of symmetries.

2.5 Task

A semi-regular tessellation is an arrangement of multiple regular polygons on the plane, such that all vertices follow the same pattern, and there are no gaps or overlaps between polygons. How many semi-regular tessellations exist? Can you prove it?

Think back to the symmetries you noticed in the regular tessellations. Are they the same here? Are there new symmetries? Does each semi-regular tessellation have the same symmetries?

2.6 Explanation

There are eleven semi-regular tessellations. One way to approach this problem is to look for all possible combinations of regular polygons whose angles add up to 360 degrees. Then play with these combinations to see which can be extended into tessellations.

There are many variations in symmetry groups. Some semi-regular tessellations fall into the same wallpaper groups as the regular tessellations, but others are different.

2.7 Task

A semi-regular polyhedron is a 3D solid formed from multiple regular polygons, such that all vertices have identical vertex patterns. How many semi-regular polyhedra can you find? Can you prove that this is all of them?

2.8 Explanation

The simplest way to prove this is to use Euler's formula:

$$V + F = E + 2 \tag{1}$$

where V represents vertices, F represents faces, and E represents edges. A complete proof can be found in A. Gardiner's *Mathematical Puzzling*.

You can look for symmetries of semi-regular polyhedra as well, although calculating the number of symmetries may be a bit overwhelming as a exercise. One interesting result worth noting, however, is that the symmetry group of the buckyball (i.e. soccer ball) is the alternating group A_5.

2.9 Task

So far, we have been talking about tiling the plane and solids. However, there is no reason to limit our exploration to these parameters. A regular tiling of 3D space fills space with exactly one type of regular polyhedra, leaving identical vertex patterns and no gaps. Similarly, a semi-regular tiling of 3D space fulfills the same criteria, but allows for the use of multiple types of regular polyhedra. The most simple regular tiling of space is a stack of cubes? Are there any others? How many semi-regular tilings of space can you find?

2.10 Explanation

Cubes form a regular tiling of space because the angle between the faces of a cube evenly divides 360. Is this true for any of the other regular polyhedra? What is the angle between the faces of a tetrahedron? Once you know the angles between the faces of all five regular polyhedra, you can easily find candidate combinations that might be able to tile space. From here, it is a simple matter of guess and check.

3 Further Directions

We have discussed this idea of a symmetry group throughout this lesson. Explain what exactly a group is give some simple examples of groups (e.g. the alternating group, the dihedral group, the cyclic group). See if you can find similarities between the symmetry groups we've encountered and these groups. What other types of symmetry groups could exist that we didn't see in the lesson?

4 References

Gardiner, A. *Mathematical Puzzling.* Birmingham, Oxford University Press, 1987.