Pick's Theorem

Adapted from A. Gardiner's *Mathematical Puzzling* by Abel Romer

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1 Organization

The goal of this lesson is to build up an intuitive understanding of Pick's Theorem and explore potential variations on Pick's original formula. Pick's Theorem states that the area A of any polygon drawn on a square lattice can be described as a simple function of of the number of dots P on the polygon's perimeter and the number of dots I contained within the polygon, such that

$$A = I + \frac{1}{2}P - 1 \tag{1}$$

Each section of this lesson will provide sequential tasks that drive at the derivation of this equation, and provide some simple leading questions to guide students/teachers/whomever in this direction. It will also include explanations of the relevant mathematics, and interesting additional directions and/or problems to tackle.

2 Lesson

2.1 Task

Suppose you have a board with pegs hammered into it in a square lattice and a rubber band to place around these pegs. How might you easily calculate the area contained within the rubber band? What techniques might you use? Calculate the areas of different shapes (e.g. a rectangle, various triangles, some sort of wonky polygon, etc.) without measuring. What did you do? What worked well? What didn't work well?

2.2 Explanation

Possible techniques for calculating areas might include: dividing irregular polygons into a number of triangles with easily determinable areas; cutting and pasting, as it were, pieces of the original shape to form one that is easier to deal with; deforming the polygon without changing the area.

2.3 Task

You have been trying to find the area of various shapes on your peg board. You probably noticed that some shapes were easier to work with than others. In particular, triangles and rectangles were especially simple. Why? Ideally, you would break a shape down into lots of little squares and then just add them all up to find the area. Unfortunately, only the simplest shapes actually allow us to do this. Is there any shape that your polygon *can* be subdivided into?

Now that you've managed to subdivide your polygon, you might want to just add up all of the little shapes. Do they all have the same area? Is there a way to ensure that they do have the same area?

2.4 Explanation

You can divide any n-sided polygon into $n-2$ of triangles. An intuition for why this is true can be developed by connecting vertices to form triangles. Each of these triangles can then be subdivided into smaller triangles by connecting edge and interior dots, until all triangles have no edge and no interior dots. These small triangles all have an area of $\frac{1}{2}$. An explanation is given below. We show:

$$P = 3 \textit{ and } I = 0 \implies A = \frac{1}{2} \tag{2}$$

$$A = \frac{1}{2} \implies P = 3 \textit{ and } I = 0 \tag{3}$$

(2) The idea here is to progressively reduce the size and complexity of the triangle while maintaining the same base and height (and therefore area). Suppose the triangle starts on points $A = (0,0)$, $B = (u,v)$ and $C = (x,y)$. Suppose $\overline{AB}$ is the shortest side. Let $\overline{AB}$ be the base. Now, as long as

we shift C along a line parallel to $\overline{AB}$, the height of the triangle will remain constant. Therefore, we shift C to a new point $D = (x-u, y-v)$. This leaves both $\overline{AB}$ and $\overline{CD}$ with slopes of $\frac{v}{u}$, thus preserving the height. We continue this procedure, now selecting the smallest side of $\triangle ABD$ to be the base. Each successive iteration will necessarily shorten the sides of the triangle, so eventually we must arrive at the smallest possible triangle, located at points $X = (a, b)$, $Y = (a+1, b)$ and $Z = (a, b+1)$. This clearly has area $A = \frac{1}{2}$. Note, if this were not a special triangle, then it would be possible for two or three points to overlap.

(3) Assume there exists a triangle with area $A = \frac{1}{2}$ that contains a dot in its perimeter or interior. Then that dot can be used to create multiple sub-triangles with $P = 3$ and $I = 0$. But we know that these triangles have an area of $\frac{1}{2}$. Since they are strictly smaller than our original triangle, our original assumption must have been false.

2.5 Task

How can you use what you have discovered about subdividing your polygon to help find its area? Can you find an expression for its area in terms of the number of little triangles? How do you know how many little triangles there are? What does the number of dots inside the polygon and on its perimeter tell you about the number of triangles?

2.6 Explanation

One way we can derive Pick's Theorem is by solving two equations for the total sum of angles S in a given polygon X. First, suppose P has some area a. Then the number of special triangles is $2a$, and so the sum is

$$\begin{aligned} S &= 2a \times 180 && (4) \\ &= 360a && (5) \end{aligned}$$

Now, we will find S a slightly different way. We know that all dots I inside of X contribute 360 degrees. Similarly, all edge dots E contribute 180 degrees. A polygon with n sides has a total degree of $(n-2) \times 180$, so all vertex dots V contribute a total of $(V-2) \times 180$ degrees. Together, this leaves us with

$$S = 360 \times I + 180 \times E + (V - 2) \times 180 \tag{6}$$

Setting these two equations equal and solving for a:

$$360a = 360 \times I + 180 \times E + (V - 2) \times 180 \tag{7}$$
$$a = I + \frac{1}{2}(E + V) - 1 \tag{8}$$

Finally, if we let $P = E + V$, we are left with

$$A = I + \frac{1}{2}P - 1 \tag{9}$$

3 Further Problems

Suppose there are holes in a polygon. Is there any easy way to calculate its area without just subtracting the area of the hole from the total area?

What about higher dimensions? Is there an equivalent formula for 3D? What about 4D?

Pick's Theorem focuses on polygons whose vertices are placed on dots. What if we remove this constraint and look at how many dots a regular polygon can contain? What is the maximum number of dots that a regular polygon with k sides of length n can hold? Does this number depend on n and k?

4 References

Gardiner, A. *Mathematical Puzzling.* Birmingham, Oxford University Press, 1987.

Johnson, Aimee S. and Joshua M. Sablof. *Pick's Theorem.* Pgs. 1-4. July 30, 2013.