List Colorings

Abel Romer

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Abstract

We introduce the notion of *list colorings* and the list chromatic number $\chi_l(G)$. We provide some general properties of $\chi_l(G)$, and then explore its value in planar and bipartite graphs. We look at a characterization by Rubin et al. of graphs with list chromatic number ≤ 2. Finally, we briefly explore the related notion of list edge chromatic number, and present the list coloring conjecture and related results.

1 Introduction

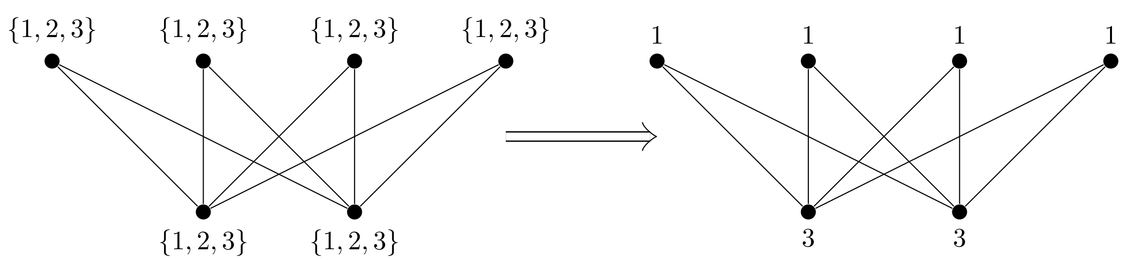
Coloring is a very common process in graph theory. Typical examples of colorings on graphs G include vertex coloring and the chromatic number $\chi(G)$, and edge coloring and the edge chromatic number $\chi'(G)$. In this paper, we explore a third type of coloring, called *list coloring*.

Definition 1.1. For a graph G, a *list coloring* of G is a proper coloring of the vertices of G from color lists $L(v)$ available at each vertex. In formal terms, let A be a set of colors, and let $L(v)$ be some subset of A assigned to vertex v. A list coloring is a function f from $L(v)$ to v such that $f(v) \neq f(u)$ for any two adjacent vertices u, v.

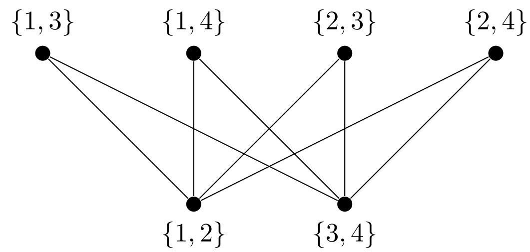
As is the case with other colorings, we would like to know the *minimum* list coloring for a given graph. We call a graph *k-choosable* if there exists a proper coloring for any list of k colors assigned to each vertex. This gives us the following definition:

Definition 1.2. The list chromatic number (or choice number) of a graph G, denoted $\chi_l(G)$, is the minimum k for which G is k-choosable.

Consider the following example on a $K_{4,2}$:



Clearly, this particular list assignment allows us to properly color $K_{4,2}$. However, we would like to know if this holds for *any* lists of size 3. Note that because we have 3 colors available, no matter what we color the bottom two vertices, we will always have at least 1 additional color for the top four. Therefore, $\chi_l(K_{4,2}) \leq 3$. Is this the best we can do? Consider the following assignment of lists of size 2:



Clearly, it is not possible to properly color this graph. No matter which two colors we select for the bottom two vertices, there exists a top vertex that must be colored using one of these two colors. Therefore, we have that $\chi_l(K_{4,2}) = 3$. This is perhaps somewhat surprising, since $\chi(K_{4,2}) = 2$. One might wonder, then, what the relation is between $\chi(G)$ and $\chi_l(G)$. We will consider this question in the following section.

Other questions of interest include:

- What is $\chi_l(G)$ for different classes of graphs? Bipartite graphs, planar graphs?
- What types of graphs have list chromatic number ≤ 2?
- What about list-edge-coloring? What are some properties of the list-edge-coloring number $\chi'_l(G)$?

2 Characterizations of $\chi_l(G)$

Motivated by the earlier example, we would like to determine the relationship of $\chi_l(G)$ to $\chi(G)$. We make the following claim:

Proposition 2.1. $\chi_l(G) \geq \chi(G)$.

Proof: Suppose $\chi(G) = k$. Let $L(v) = \{1, 2, ..., k\}$ for all $v \in V(G)$. These lists admit a proper k-coloring of G. Furthermore, if $L(v) = \{1, 2, ..., k-1\}$ for all $v \in V(G)$, then G clearly cannot be properly colored. Therefore, $\chi_l(G) \geq \chi(G)$.

We present three further properties of $\chi_l(G)$, courtesy of Erdos et al. [1], Thomassen [2], and Alon et al. [3], respectively:

1. $\chi_l(G) \leq \Delta(G) + 1$.
2. $\chi_l(G) \leq 5$ for planar G.
3. $\chi_l(G) \leq 3$ for bipartite planar G.

To gain an intuition for (1), note that no matter what colors are assigned to a given vertex, it will always be possible to properly color that vertex and all its neighbors, even if all its neighbors are connected.

We will explore (2) and (3) in the following sections.

3 Graph classes

3.1 Bipartite graphs

Since bipartite graphs provide a strict bound on $\chi(G)$ (namely, $\chi(G) = 2$), it is natural to ask whether a similar statement holds for $\chi_l(G)$. Unfortunately, we shall find that this is simply not the case. In our earlier example, we observed that $\chi_l(K_{4,2}) = 3$ by providing a vertex list assignment of size 2 for which $K_{4,2}$ was not properly colorable. We now generalize this technique across all complete bipartite graphs $K_{n^n,n}$.

Proposition 3.1. *For complete bipartite graphs G of the form $K_{n^n,n}$, $\chi_l(G) \geq n+1$.*

Proof: Let $G = (X, Y)$, where $|X| = n^n$ and $|Y| = n$. We provide a vertex list assignment of size n such that G is not properly colorable. For $u_i \in Y$, let $L(u_i) = \{i1, i2, ..., in\}$. For $v_i \in X$, let $L(v_i)$ be one of the n^n possible sets made from taking one color from each $L(u_i)$. Then, no matter what colors we select for the vertices of Y, there will necessarily be some vertex $v \in X$ such that $L(v)$ is exactly the colors of the vertices in Y.

A similar idea gives us the following theorem, due to Erdos et al. [1]:

Theorem 3.2. *For all complete bipartite graphs G of the form $K_{n,n}$, where $n = \binom{2k-1}{k}$, $\chi_l(G) > k$.*

Proof: We show that $G = (X, Y)$ cannot be properly colored using lists of size k. Let there be $2k - 1$ available colors (i.e. $|A| = 2k - 1$). Consider the n unique subsets of A of size k. To each vertex in X and Y, assign one such unique subset. We have two scenarios:

1. We see that in our coloring of X, we must use at least k colors. Otherwise, the selected colors would be disjoint from the list of colors available to one vertex in X, meaning that vertex would be uncolorable.

2. However, if we use k colors for the vertices of X, then there will be some vertex in Y whose list includes exactly those k colors.

No matter what we do, we see that a k-coloring is not possible. We therefore conclude that G is not k-choosable.

We present one final characterization of $\chi_l(G)$ for bipartite G, due to Alon et al. [3]:

Theorem 3.3. *Let $L(G) = max(|E(H)|/|V(H)|)$ for all subgraphs H of G. Then every bipartite graph G is $(\lceil L(G) \rceil + 1)$-choosable.*

For a detailed proof of this result, we direct the reader to the original paper by Alon et al. [3].

3.2 Planar graphs

As we alluded to earlier, Thomassen [2] discovered a nice upper bound for the list chromatic number of planar graphs. Namely:

Theorem 3.4. *For planar G, $\chi_l(G) \leq 5$.*

Proof: Suppose G is maximally planar (i.e. all bounded faces are triangles, and the outer face is a cycle C). We show that if G has two adjacent external vertices with different lists of size 1, all other external vertices have lists of size 3, and all internal vertices have lists of size 5, then G can be properly colored. (Note that this claim is stronger than the theorem.) Since adding edges to a graph cannot reduce its list chromatic number, it is sufficient to prove this claim for maximal G. We perform induction on $n(G)$:

Base case: If $n = 3$, then G is a triangle. Let $V(G) = \{v_1, v_2, v_3\}$, and $L(v_1) = \{1\}$ and $L(v_2) = \{2\}$. Then, since $|L(v_3)| = 3$, there exists a color for v_3 that is different from v_1 and v_2.

Induction step: We consider graphs on > 3 vertices. Let $C = v_1v_2v_3...v_pv_1$ be the external cycle of G. Suppose that v_1 and v_p have different color lists both of size 1. We consider two cases:

1. Suppose G has a chord between vertices v_i and v_j in C. We consider the two smaller cycles $C' = v_1v_2...v_iv_j...v_pv_1$ and $C'' = v_iv_{i+1}...v_jv_i$. We first apply our induction hypothesis to C' and its interior. This provides a proper coloring where v_i, v_j are adjacent and differently colored. We then apply our induction hypothesis to C'' and its interior. The resulting colorings give us a list coloring of G.

2. Suppose G does not have a chord. Let $\{v_1, u_1, ..., u_m, v_3\}$ be the neighborhood of v_2. Since G is maximally planar, $v_1u_1...u_mv_3$ must form a path from v_1 to v_m. Furthermore, since G is chordless, $\{u_1, ..., u_m\}$ are all internal vertices of G. We consider the graph $G' = G - v_2$.

   Let v_1 be colored with color c. Since $|L(v_2)| = 3$, there exist two colors $x, y \in |L(v_2)|$ such that $x \neq c$ and $y \neq c$. For all $u_i \in \{u_1, ..., u_m\}$, let $L'(u_i) = L(u_i) \setminus \{x, y\}$. This is possible since all such vertices are internal and therefore have lists of size 5. Therefore, $|L'(u_i)| \geq 3$. We now apply the induction hypothesis to G'. This provides a proper coloring for all vertices of G except v_2. However, since v_3 is the only neighbor of v_2 that can be colored one of x or y, we simply color v_3 whichever of these colors is not assigned to v_2.

Additionally, a result from Alon et al. [3] shows that if G is both planar and bipartite, then this upper bound is even tighter:

Theorem 3.5. *For bipartite planar G, $\chi_l(G) \leq 3$.*

Proof: This follows almost directly from theorem 3.3. We observe that if G is bipartite and planar, then $L(G) \leq 2$. This is because any simple bipartite planar graph on n vertices can have at most $2n - 4$ edges. Therefore, G is $(2+1) = 3$-choosable.

4 What if $\chi_l(G) \leq 2$?

In this section, we will attempt to characterize all graphs that are at most 2-choosable. Most of the results in the section are due to a 1979 paper by Paul Erdős, Arthur Rubin, and Herbert Taylor [1].

We begin by observing that any vertices of degree 1 can be added to a 2-choosable graph without changing its list chromatic number. We therefore only consider those graphs with minimum degree ≥ 2. Let the *core* of a graph G be the graph resulting from the deletion of all degree 1 vertices from G. We define the following class of graphs:

Definition 4.1. We call a graph G a Θ-graph if G consists of two vertices u and v with three vertex-disjoint paths between them.

Note that a given Θ-graph can be characterized by the lengths of these three paths. For example, a C_4 with a chord is the Θ-graph given by $\Theta_{2,2,1}$.

We then have the following characterization of 2-choosable graphs:

Theorem 4.1. *Let $T = \{K_1, C_{2m+2}, \Theta_{2,2,2m} : m \geq 1\}$. Then G is 2-choosable if and only if the core of G is in T.*

A detailed examination of this theorem can be found in [1]. In this paper, we will simply provide a proof of the 2-choosability of the elements of T. Consider $\Theta_{2,2,2m}$. Let the vertices on paths of length 2 between u and v be A and B. We consider two cases:

1. Suppose that all $2m$ vertices from u to v are assigned the same color lists $\{x, y\}$. Then we may color these vertices by alternating between x and y. Since there are $2m$ such vertices, $c(u) = c(v) = x$. Color the remaining two disjoint vertices A, B between u and v with y.

2. Suppose that the $2m$ vertices from u to v are assigned different color lists. Select two adjacent such vertices v_i, v_j. Proceed towards v, properly coloring vertices along this path as you go. Suppose that you color v with some color x. Now consider $L(u) = \{a, b\}$. If $\{A, B\} \neq \{\{x, a\}, \{x, b\}\}$, then color A and B with a and b, respectively, and continue along until you arrive back at v_i, thereby completing a proper coloring. If, however, $\{A, B\} = \{\{x, a\}, \{x, b\}\}$, return back to v_i, v_j, and repeat the algorithm, this time heading in the direction of u. Since we know $c(u) \neq x$, we can color A and B with x, and then proceed happily along back to v_j, thereby completing a proper coloring.

Note that since $\Theta_{2,2,2m}$ is 2-choosable, and C_{2m+2} is a subgraph of $\Theta_{2,2,2m}$, C_{2m+2} must also be 2-choosable.

5 List-edge-coloring

We examine a natural extension of the list-chromatic number of a graph: the list-edge-chromatic number $\chi'_l(G)$.

Definition 5.1. A *list-edge-coloring* of a graph G is a proper coloring of the edges of G from color lists $L(e)$ available at each edge. A graph is k-edge-choosable if there exists a proper coloring for any list of k colors assigned to each edge. The list-edge-chromatic number of a graph G, denoted $\chi'_l(G)$, is the minimum k for which G is k-edge-choosable.

As with the list-chromatic number, we would like to find ways to characterize and/or bound $\chi'_l(G)$. However, unlike $\chi_l(G)$, which differs significantly from $\chi(G)$, the list-edge-chromatic number and the edge-chromatic number are quite similar. In fact, it is conjectured that the two values may be equal. Some properties of $\chi'_l(G)$ follow:

1. $\chi'_l(G) < 2\chi'(G)$.
2. $\chi'_l(G) < (1 + o(1))\chi'(G)$.
3. $\chi'_l(K_{n,n}) = n$.

The first property follows from the fact that $\chi'(G) \leq 2\Delta(G) - 1$ and $\chi'_l(G) \leq 2\Delta(G) - 1$. The second property is a result of a 2000 paper by Jeff Kahn [5], and states that the list-edge-chromatic number and the edge-chromatic number are asymptotically equal. The final property is due to Galvin [4], and is discussed in greater detail below.

5.1 List coloring conjecture

As suggested in the previous section, the list coloring conjecture is as follows:

Conjecture 5.1. For all graphs G, $\chi'_l(G) = \chi'(G)$.

Incremental progress has been made on this problem, although there is as of yet no known solution. As mentioned above, the two values have been shown to be asymptotically equal [5], and there are papers addressing specific cases of the list coloring conjecture in small complete graphs [6], perfect multigraphs [7], and complete bipartite graphs [4]. We shall conclude this paper with a brief discussion of the list coloring conjecture for complete bipartite graphs.

5.2 Galvin's theorem

Galvin's theorem is a specific instance of the list coloring conjecture for complete bipartite graphs $K_{n,n}$. It states the following:

Theorem 5.1. *For all complete graphs $K_{n,n}$, $\chi'_l(K_{n,n}) = n$.*

Prior to its proof in 1995, this statement was known as the Dinitz Conjecture. It was typically formulated as a question about Latin squares; namely, given an $n \times n$ square array, a set of $m \geq n$ symbols, and an n-element list of symbols available to each cell, is it possible to fill in the grid such that no column and no row contain repeated symbols? We will provide a brief outline of the proof of this theorem. For a detailed examination, see [4].

The proof combines two previously known results. The first is a lemma due to Bondy et al., which states that for any digraph G with vertices $\{v_1, v_2, ..., v_n\}$ such that the out-degree of v_i is d_i, and such that every induced subgraph of G has a kernel, there exists a set A_i of size at least $d_i + 1$ such that each vertex v_i of G may be properly colored with some color from A_i. The second is a version of the Gale-Shapley algorithm, which states that given two sets, each with preferences for each other, it is always possible to find a pairing that is stable. Galvin found a clever way to combine these two results, along with properties of complete bipartite graphs, to show that the edges of $K_{n,n}$ can always be properly colored when assigned lists of size n.

6 References

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