Supersaturation and Generalizations of Sperner's Theorem*

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1 Introduction

The field of extremal combinatorics deals with questions of maximization and minimization, subject to particular constraints. These questions are often presented in the context of sets (extremal set theory) and graphs (extremal graph theory). Perhaps the most fundamental questions of extremal set theory and extremal graph theory are, respectively:

Question 1.1. What is the largest family of sets, such that no set contains any other?

Question 1.2. What triangle-free graph has the most edges?

In this introduction, we present a brief overview of results relating to these questions. We examine some natural extensions of these results, including the concept of supersaturation. In the following section, we explore the problem of supersaturation in set theory, and present a result by Dove et al. [1] regarding the supersaturation of chains. We conclude the paper by reviewing some other supersaturation-type results and proposing avenues of further study.

1.1 Antichains and Sperner's Theorem

We aim to determine the cardinality of the largest collection of sets, where no set contains any other. To frame this problem formally, we define the following:

Definition 1.1. A family $\mathcal{A} \subseteq 2^{[n]}$ is an *antichain* if, for all sets $A, B \in \mathcal{F}$, $A \nsubseteq B$ and $B \nsubseteq A$.

Question 1.1 therefore asks: what is the size of the largest antichain $\mathcal{A} \subseteq 2^{[n]}$? This problem was first solved by Emanuel Sperner in 1928, and the result is now known as Sperner's Theorem [2]. It states:

Theorem 1.1 (Sperner)**.** *The largest size of an antichain* $\mathcal{A} \subseteq 2^{[n]}$ *is*

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

A number of proofs of Sperner's Theorem are known. We will present a sketch of a proof involving a technique known as "symmetric chain decomposition," or SCD. We first introduce three concepts:

Definition 1.2. Let a *chain* $\mathcal{C} \subseteq 2^{[n]}$ be a family of sets where for every pair $A, B \in \mathcal{C}$, either $A \subseteq B$ or $B \subseteq A$.

Definition 1.3. Let a *symmetric* chain be a chain that contains sets of every cardinality i for $i \in \{k, k+1, \ldots, n-k\}$, $0 \leq k \leq n/2$.

Definition 1.4. Let the *symmetric chain decomposition* of a collection $\mathcal{F}$ be a disjoint partition of the elements of $\mathcal{F}$ into symmetric chains.

*This paper is a summary of *Supersaturation in the Boolean lattice* by Dove et al. [1]

Proof idea for Theorem 1.1. Show that $2^{[n]}$ has a symmetric chain decomposition $\mathcal{C}_1, \ldots, \mathcal{C}_N$. By the definition of the SCD, each $\mathcal{C}_i$ contains exactly one element of the set $\binom{[n]}{\lfloor n/2 \rfloor}$, and so

$$N = \binom{n}{\lfloor n/2 \rfloor}.$$

Since the intersection of a chain and an antichain can be at most 1, the cardinality of some antichain $\mathcal{A} \subseteq 2^{[n]}$ is at most $\binom{n}{\lfloor n/2 \rfloor}$. □

There are a number of natural extensions to Sperner's Theorem. Perhaps the most obvious generalizes the question from antichains to collections that forbid chains of length k. To explore this problem, we introduce some additional notation:

Definition 1.5. Let the *Boolean lattice* be the partially ordered set $\mathscr{B}_n = (2^{[n]}, \subseteq)$ of all subsets of $[n]$, ordered by inclusion.

Definition 1.6. Let $\mathscr{P}_k$ refer to the totally ordered poset (or chain) of k sets.

Definition 1.7. For two posets $(P, \preceq)$ and $(P', \preceq')$, let P' *contain* P if there exists an injection from P to P' that preserves the partial order.

It is common to think of the Boolean lattice as the set of all binary strings of length n (hence the name Boolean), where a 1 refers to inclusion of the ith element of $[n]$, and a 0 refers to exclusion. A diagram of $\mathscr{B}_3$ is shown in figure 1. Note that the collections $\{\varnothing, \{1\}, \{1,2\}\}$ and $\{\varnothing, \{1,2\}, \{1,2,3\}\}$ are both $\mathscr{P}_3$: totally

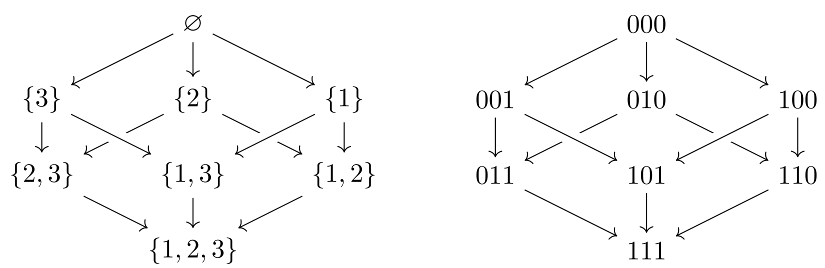

Figure 1: A Hasse diagram of the Boolean lattice $\mathscr{B}_3$. The left diagram uses set inclusion, and the right uses binary strings. Arrows indicate inclusion (e.g. 001 can be considered a subset of 011).

ordered collections of cardinality 3. Also, note that if we let $P' = \{\varnothing, \{1\}, \{1,2\}\}$ and $P = \{\varnothing, \{1,2\}\}$, then P' contains P.

With these definitions under our belt, we are equipped to address a more generalized version of Sperner's Theorem. Specifically, we would like to know: what is the largest size of a collection of subsets of $[n]$ that does not contain some partially ordered set P? We shall refer to this number as $\mathrm{La}(n, P)$. Applying our new notation, Sperner's Theorem states:

$$\mathrm{La}(n, \mathscr{P}_2) = \binom{n}{\lfloor n/2 \rfloor}.$$

The tightness of this bound is witnessed by selecting the $\binom{n}{\lfloor n/2 \rfloor}$ elements in the middle row of the Boolean lattice (in our example, this is either the set $\{\{1\}, \{2\}, \{3\}\}$ or the set $\{\{2,3\}, \{1,3\}, \{1,2\}\}$). We shall see constructions involving the "middle rows" of the Boolean lattice a great deal in this paper. Therefore, for ease of notation, we shall use $\mathscr{B}(n,k)$ to refer to the k middle rows of the Boolean lattice $\mathscr{B}_n$, and $\Sigma(n,k)$ to refer to the cardinality of $\mathscr{B}(n,k)$.

It turns out that the more general problem of determining $\mathrm{La}(n, \mathscr{P}_k)$ is also known. This result was discovered by Erdös roughly two decades after Sperner's proof of the $\mathscr{P}_2$ case [3]. It states:

$$\mathrm{La}(n, \mathscr{P}_k) = \Sigma(n, k-1).$$

Again, we shall forego a proof of this result, but note that the collection $\mathscr{B}(n, k-1)$ witnesses this bound.

Finally, we can generalize beyond chains, and simply attempt to determine $\text{La}(n, P)$ for arbitrary P. This is a particularly challenging problem (and in many senses is quite analogous to the problem of determining the extremal number of arbitrary graphs, which we shall review in the following section). Katona is responsible for much of the progress in this area, and some of his results are presented in [4].

1.2 The extremal number of graphs

In much the same way that we have attempted to determine the largest family of sets that does not contain a particular poset P, we might also try to determine the maximum number of edges in a graph G such that G that does not contain some smaller graph H. We refer to this value as the *extremal number* of H. Formally, we define:

Definition 1.8. Let the *extremal number* $ex(n, H)$ of a graph H be the maximum number of edges in an n-vertex graph G such that G does not contain a copy of H.

Much akin to Sperner's Theorem for sets, there exists a result from Mantel that gives $ex(n, K_3)$—that is, the maximum number of edges in a triangle-free graph [5].

Theorem 1.2 (Mantel)**.** *The maximum number of edges in a triangle-free graph is given by*

$$ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

The proof of this result is outside of the scope of this paper; however, the construction that maximizes the number of edges in a triangle-free graph is a complete bipartite graph with partitions of roughly (within the floor/ceiling) equal size.

In the same way that $\text{La}(n, \mathscr{P}_k)$ generalizes somewhat readily from $\text{La}(n, \mathscr{P}_2)$, Mantel's Theorem has been generalized by Turàn to cover all complete graphs K_r. Specifically, Turàn found that the complete $(r-1)$-partite graph with partitions of roughly (with floor/ceiling) equal size maximizes the number of edges in a K_r-free graph [6].

Continuing the analogy to $\text{La}(n, P)$, the problem of determining $ex(n, H)$ for arbitrary H remains a difficult and much-studied subject in extremal graph theory. Some additional results are known, but the subject remains far from closed.

1.3 Supersaturation

In the previous sections, we have looked at some extensions of two classic problems in the fields of extremal set theory and extremal graph theory. We now examine a different generalization. Rather than avoiding some structure (e.g. a poset P or a graph H), we might ask: what is the minimum number of copies of H (or P) in a graph (or collection) with $x + ex(n, H)$ edges ($\text{La}(n, P)$ sets)? What is the "best" way to add edges (sets) to G ($\mathcal{F}$) so that we minimize the number of copies of H (or P)? This is the process of supersaturation.

We begin with graphs:

Definition 1.9. Let $\ell(n, H, q)$ be the minimum number of copies of H in an n-vertex graph with at least $ex(n, H) + q$ edges.

We would like to determine the value of $\ell(n, H, q)$ for different choices of H and q. As we have done previously, we shall start with the simplest case; namely, we will aim to find $\ell(n, K_3, q)$. This problem was first introduced by Rademacher in the early 1940s. The first published result from Erdős in 1955 showed that $\ell(n, H, q) \geq \lfloor n/2 \rfloor$ for $q \in \{1, 2, 3\}$. Nearly 30 years later, Lovasz and Simonovits generalized this result into the following theorem [7]:

Theorem 1.3. *For positive integers n and q, if $q < n/2$, then $\ell(n, K_3, q) \geq q \cdot \lfloor n/2 \rfloor$.*

	Extremal Problems			Supersaturation		
	$K_3/\mathscr{P}_2$	$K_r/\mathscr{P}_k$	H/P	$K_3/\mathscr{P}_2$	$K_r/\mathscr{P}_k$	H/P
Graphs	yes	yes	HARD	yes	yes	HARD
Sets	yes	yes	HARD	???	???	???

Table 1: A summary of known results for extremal problems in set and graph theory.

This result tells us that every edge added to a graph with the extremal number of edges will form $\lfloor n/2 \rfloor$ new triangles. Further developments have been made in recent years, with results for $\ell(n, K_4, q)$ from Nikiforov [8], and $\ell(n, K_r, q)$ from Reiher [9].

Up to this point, we have seen a somewhat natural progression of results related to extremal problems in set theory and graph theory. These are summarized in table 1. Based on the symmetry of this table, we might expect that the problem of supersaturation of chains should be (somewhat) readily solvable. The following section is dedicated to this result.

2 Supersaturation in the Boolean Lattice

The problem of supersaturation in sets involves determining the minimum number of occurrences of some poset P in a family of size $x + \mathrm{La}(n, P)$. We shall examine the specific case where $P = \mathscr{P}_k$. Recall that in the case of chains, we have $\mathrm{La}(n, \mathscr{P}_k) = \Sigma(n, k-1)$.

We present a proof from Dove et al. [1] that establishes a lower bound on the number of k-chains contained in a family $\mathcal{F}$ of size $x + \Sigma(n, k-1)$. This result is achieved by a clever application of symmetric chain decomposition of the Boolean lattice. We proceed as follows:

1. We determine a weak lower bound on the number of poset SCDs that contain a given k-chain of $\mathcal{F}$ (which we refer to as $N(n, A_1, \ldots, A_k)$).

2. We use the result from (1) to find a weak lower bound on the number of appearances of each k-chain of $\mathcal{F}$ in all possible poset SCDs of $\mathscr{B}_n$. This step allows us to accommodate for the fact that certain poset SCDs may not contain particular k-chains. In other words, we wish to remain unbiased in our selection of a poset SCD. We will ultimately divide through by the number of poset SCDs to obtain an average.

3. We upper bound $N(n, A_1, \ldots, A_k)$ using some clever algebraic techniques.

4. We use the upper bound in (3) to obtain a stronger lower bound on the number of k-chains of $\mathcal{F}$.

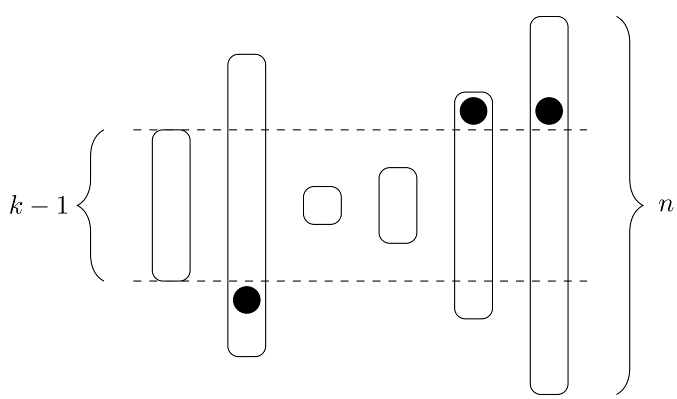

Figure 2: An "artist's rendition" of a symmetric chain decomposition $\mathscr{C}$ of $\mathscr{B}_n$. The area between the dashed lines represents $\mathscr{B}(n, k-1)$. The dots represent those sets corresponding to x in $x + \Sigma(n, k-1)$. Note that every dot forms at least one chain of length k.

Theorem 2.1. *Let $\mathcal{F}$ be a family of subsets of $[n]$, where $|\mathcal{F}| = x + \Sigma(n, k-1)$. Then $\mathcal{F}$ contains at least*

$$x \cdot \prod_{i=1}^{k-1} \left(\left\lfloor \frac{n+k}{2} \right\rfloor - i + 1 \right)$$

copies of $\mathscr{P}_k$.

Proof. We proceed according to the steps laid out above:

1. Let $\mathscr{C}$ be an arbitrary symmetric chain decomposition of $\mathscr{B}_n$, and let $c_{\mathcal{F}}(P)$ be the number of k-chains of $\mathcal{F}$ in some poset P. Let $P_{\mathscr{C}}$ be the poset composed of the disjoint union of chains in $\mathscr{C}$. We wish to determine a lower bound on $c_{\mathcal{F}}(P_{\mathscr{C}})$. Since, by definition, $|\mathcal{F}| = x + \Sigma(n, k-1)$, we have that $c_{\mathcal{F}}(P_{\mathscr{C}}) \geq x$ (see figure 2).

2. We extend the result from (1) to determine a lower bound on the number of k-chains of $\mathcal{F}$ in *all* SCDs of $\mathscr{B}_n$. Let π be a permutation of the set $[n]$. Note that for any SCD $\mathscr{C}$, $\pi(\mathscr{C})$ is also an SCD. Therefore, summing over all $n!$ permutations gives:

$$\sum_{\pi} c_{\mathcal{F}}(P_{\pi(\mathscr{C})}) \geq n! \cdot x.$$

   Rather than sum over all permutations π (as we have done above), we might equivalently sum over all of the k-chains in $\mathcal{F}$. (The benefit of this choice, as we shall see later, is that it will allow us to solve the resulting inequality for the number of k-chains in $\mathcal{F}$.) In this case, the terms in our sum count the number of distinct permutations that contain a particular k-chain $A_1 \subset \cdots \subset A_k$, which we represent by $N(n, A_1, \ldots, A_k)$. Putting all this together, we have:

$$\sum_{\substack{A_1 \subset \cdots \subset A_k \\ A_i \in \mathcal{F}}} N(n, A_1, \ldots, A_k) = \sum_{\pi} c_{\mathcal{F}}(P_{\pi(\mathscr{C})}) \geq n! \cdot x.$$

3. We now determine an upper bound on $N(n, A_1, \ldots, A_k)$. Recall that this expression counts the number of posets $P_{\pi(\mathscr{C})}$ that contain a given k-chain $A_1 \subset \cdots \subset A_k$. Let $a_i = |A_i|$ for $1 \leq i \leq k$. We claim that there are

$$a_1! \cdot (a_2 - a_1)! \cdot (a_3 - a_2)! \cdots (a_k - a_{k-1})! \cdot (n - a_k)! \cdot \min\left\{ \binom{n}{a_1}, \binom{n}{a_k} \right\}$$

   possible symmetric chains that can contain $A_1 \subset \cdots \subset A_k$. Note that each of the terms $(a_i - a_{i-1})$ counts the number of sets in a symmetric chain between A_i and A_{i-1}. Similarly, a_1 counts the number of sets preceding A_1 (from $\varnothing$ to A_1), and $(n - a_k)$ counts the number of sets following A_k (from A_k to $[n]$). Taking the factorial of these terms counts all possible combinations of these "missing" sets. Therefore, the expression

$$a_1! \cdot (a_2 - a_1)! \cdot (a_3 - a_2)! \cdots (a_k - a_{k-1})! \cdot (n - a_k)!$$

   counts the number of complete (from $\varnothing$ to $[n]$) symmetric chains that contain a given k-chain $A_1 \subset \cdots \subset A_k$. To accommodate for the fact that not all such chains need be complete, we multiply by

$$\min\left\{ \binom{n}{a_1}, \binom{n}{a_k} \right\}.$$

   Since $\binom{n}{a_i}$ counts the number of sets in row i of $\mathscr{B}_n$, and in an SCD there is exactly one chain through each of these sets, this expression counts the number of chains in an SCD that are large enough to contain $A_1 \subset \cdots \subset A_k$. As no two chains in a given SCD can contain that same k-chain, it must be the case that

$$N(n, A_1, \ldots, A_k) = a_1! \cdot (a_2 - a_1)! \cdot (a_3 - a_2)! \cdots (a_k - a_{k-1})! \cdot (n - a_k)! \cdot \min\left\{ \binom{n}{a_1}, \binom{n}{a_k} \right\}.$$

We would like to determine an upper bound on the above expression. To do so, we determine the exact structure of k-chains that maximize its value. We claim that $N(n, A_1, \ldots, A_k)$ is maximized when $A_1 \subset \cdots \subset A_k$ is a symmetric chain itself. In other words, when we have

$$a_1, a_2, \ldots, a_{k-1}, a_k = \left\lfloor \frac{n+k}{2} \right\rfloor - (k-1), \left\lfloor \frac{n+k}{2} \right\rfloor - k, \ldots, \left\lfloor \frac{n+k}{2} \right\rfloor - 1, \left\lfloor \frac{n+k}{2} \right\rfloor.$$

This fact may seem somewhat intuitive; we shall prove it using a clever algebraic argument. Observe that the product of binomial coefficients

$$\binom{a_k}{a_{k-1}}\binom{a_{k-1}}{a_{k-2}} \cdots \binom{a_2}{a_1}$$

is, upon expanding and cancelling terms, equal to

$$\frac{a_k}{a_1! \cdot (a_2 - a_1)! \cdot (a_3 - a_2)! \cdots (a_k - a_{k-1})!}.$$

Taking the inverse and multiplying by $n!$, we see that the above expression is equal to

$$\frac{n!}{\binom{a_k}{a_{k-1}}\binom{a_{k-1}}{a_{k-2}} \cdots \binom{a_2}{a_1}} = a_1! \cdot (a_2 - a_1)! \cdot (a_3 - a_2)! \cdots (a_k - a_{k-1})! \cdot (n - a_k)! \cdot \binom{n}{a_1}.$$

Applying the same process to

$$\binom{n-a_1}{n-a_2}\binom{n-a_2}{n-a_3} \cdots \binom{n-a_{k-1}}{n-a_k},$$

we see that

$$\frac{n!}{\binom{n-a_1}{n-a_2}\binom{n-a_2}{n-a_3} \cdots \binom{n-a_{k-1}}{n-a_k}} = a_1! \cdot (a_2 - a_1)! \cdot (a_3 - a_2)! \cdots (a_k - a_{k-1})! \cdot (n - a_k)! \cdot \binom{n}{a_k}.$$

Combining these two results, we have:

$$N(n, A_1, \ldots, A_k) = \frac{n!}{\max\left\{ \binom{a_k}{a_{k-1}}\binom{a_{k-1}}{a_{k-2}} \cdots \binom{a_2}{a_1}, \binom{n-a_1}{n-a_2}\binom{n-a_2}{n-a_3} \cdots \binom{n-a_{k-1}}{n-a_k} \right\}}.$$

We first show that the above expression is maximized when the difference between all a_i, a_{i+1} is 1. Let $y = \binom{a_k}{a_{k-1}} \cdots \binom{a_2}{a_1}$ and $z = \binom{n-a_1}{n-a_2} \cdots \binom{n-a_{k-1}}{n-a_k}$, where the difference of all consecutive a_i's is 1. Suppose that there exist two consecutive a_i's with difference greater than 1. Without loss of generality, let these be a_k and a_{k-1}. Let

$$y' = y \cdot \frac{\binom{a_{k-1}+i}{a_{k-1}}}{\binom{a_{k-1}+1}{a_{k-1}}} \quad \text{and} \quad z' = z \cdot \frac{\binom{n-a_2+i}{n-a_2}}{\binom{n-a_2+1}{n-a_2}},$$

where $i \geq 2$. Since $\binom{n+1}{k} > \binom{n}{k}$, provided $k > 0$, we see that

$$y' > y \quad \text{and} \quad z' > z$$

for all $a_k > 0$ and $a_2 < n$, respectively. This tells us that y and z are minimized (and therefore $N(n, A_1, \ldots, A_k)$ is maximized) when the difference of all consecutive a_i's is 1. Supposing that this is indeed the case, we have

$$y = \frac{a_k!}{(a_k - k + 1)!} \quad \text{and} \quad z = \frac{(n - a_k + k - 1)!}{(n - a_k)!},$$

and so

$$N(n, A_1, \ldots, A_k) \leq \frac{n!}{\min\left\{\frac{a_k!}{(a_k-k+1)!}, \frac{(n-a_k+k-1)!}{(n-a_k)!}\right\}}.$$

Finally, note that the denominator

$$\min\left\{\frac{a_k!}{(a_k-k+1)!}, \frac{(n-a_k+k-1)!}{(n-a_k)!}\right\}$$

is minimized by letting $a_k = \lfloor (n+k)/2 \rfloor$. Plugging this into the above inequality and simplifying returns:

$$N(n, A_1, \ldots, A_k) \leq \frac{n!}{\prod_{i=1}^{k-1}\left(\lfloor \frac{n+k}{2} \rfloor - 1 + 1\right)}.$$

This is our upper bound on $N(n, A_1, \ldots, A_k)$.

4. Finally, we plug this upper bound into the inequality from (2). This gives:

$$n! \cdot x \leq \sum_{\substack{A_1 \subset \cdots \subset A_k \\ A_i \in \mathcal{F}}} N(n, A_1, \ldots, A_k) \leq \sum_{\substack{A_1 \subset \cdots \subset A_k \\ A_i \in \mathcal{F}}} \frac{n!}{\prod_{i=1}^{k-1}\left(\lfloor \frac{n+k}{2} \rfloor - 1 + 1\right)}.$$

Observe that the RHS is simply summing over all k-chains in $\mathcal{F}$. By our notation from (1), there are $c_{\mathcal{F}}(\mathcal{F})$ such chains. Therefore, the above inequality is equivalent to:

$$n! \cdot x \leq c_{\mathcal{F}}(\mathcal{F}) \frac{n!}{\prod_{i=1}^{k-1}\left(\lfloor \frac{n+k}{2} \rfloor - 1 + 1\right)}.$$

Solving for $c_{\mathcal{F}}(\mathcal{F})$ gives the result:

$$c_{\mathcal{F}}(\mathcal{F}) \geq x \cdot \prod_{i=1}^{k-1}\left(\left\lfloor \frac{n+k}{2} \right\rfloor - i + 1\right).$$

□

In the following section, we provide some observations regarding this proof, and propose steps that could be taken to improve the result.

2.1 Notes on the proof

First, note that the expression

$$\prod_{i=1}^{k-1}\left(\left\lfloor \frac{n+k}{2} \right\rfloor - i + 1\right)$$

counts exactly the number of copies of $\mathscr{P}_k$ in the k middle rows of the Boolean lattice $\mathscr{B}_n$ that originate in a given set in the kth middle row. Each factor $\lfloor \frac{n+k}{2} \rfloor - i + 1$ gives the ith index of each of the k middle rows. For example, if $n = 10$ and $k = 4$, we can see that the above product multiplies over factors $7, 6, 5$ and 4. Furthermore, we can see that each set in the ith row of the Boolean lattice is a superset of i sets from the row above it. Therefore, the above product counts all possible sequences of containment of length k that start from a particular set. This is precisely equal to the number of copies of $\mathscr{P}_k$ that begin with a particular set in the kth middle row.

From this observation, we note that the bound in theorem 2.1 is tight for

$$x \leq \binom{n}{\lfloor \frac{n}{2} \rfloor + (-1)^k \lfloor \frac{k}{2} \rfloor}.$$

This is witnessed by letting $\mathcal{F} = \mathscr{B}(n, k-1) + x$, where each set in x comes from the kth middle row. The idea is that each of the x sets corresponds to a different starting set for a k-chain that passes through the

k middle rows of the Boolean lattice $\mathscr{B}_n$. The upper bound on x is simply the number of possible starting sets.

These observations lead us to believe that any construction that selects sets from near the middle of the Boolean lattice will naturally minimize the number of k-chains. While the above result only proves this for sets no larger than $\Sigma(n, k)$, it seems likely that this construction holds for all $\mathcal{F}$.

3 Extensions

We conclude by asking and addressing the following questions regarding supersaturation and lattice-like structures:

a. *What about set structures? Do these results extend to other lattices?*

   There has been research focused on supersaturation in structures other than the Boolean lattice. A 2018 paper by Noel et al. [13] addresses an extension of Sperner's Theorem to arbitrary posets P. The problem involves determining the number of comparable pairs (chains of length 2) in a subset of P of size m (where m is greater than the largest antichain in P).

   Other lattice structures possibly of interest include the distributive lattice and the lattice of partitions. It is also interesting to investigate extremal problems in lattice-type structures that do not admit decomposition into symmetric chains. While it is likely that problems of this sort have been investigated, the amount of literature on the topic is limited.

b. *What about supersaturation for sets other than $\mathscr{P}_k$?*

   The problem of determining $\text{La}(n, P)$ has been studied for a large number of posets P. These include the diamond poset $\mathcal{Q}_2$ (defined as the set of four elements w, x, y, z with relations $x < y$, $x < z$, $w > y$, $w > z$), the V poset (defined as the set of three elements x, y, z with relations $x \leq y$, $x \leq z$), and the Λ poset (defined as the set of three elements x, y, z with relations $x \leq z$, $y \leq z$) [10]. A 2015 survey paper by Griggs et al. summarizes many of these results [11]. Interestingly enough, the seemingly innocuous problem of determining $\text{La}(n, \mathcal{Q}_2)$ remains open.

   There is apparently little research addressing supersaturation-type questions for these posets. It seems plausible that a supersaturation result for the V and Λ posets might follow readily from results related to $\text{La}(n, V)$ and $\text{La}(n, \Lambda)$. One might feel similarly about the diamond poset, but for the fact that $\text{La}(n, \mathcal{Q}_2)$ remains open. It is, however, unclear whether the methodology employed in Dove et al. has an analogous form for posets that are not chains. The proof presented in this paper relies heavily on SCDs of the Boolean lattice, and such techniques appear unlikely to prove useful when dealing with posets like $\mathcal{Q}_2$, V and Λ. Perhaps there are other decompositions of the Boolean lattice that afford similar benefits? Or perhaps a novel approach is necessary.

   There exists an analogous supersaturation result for the butterfly poset (defined as the set of four elements w, x, y, z with relations $a \leq c$, $a \leq d$, $b \leq c$, $b \leq d$). This result is described in a 2015 paper by Pátkos [12]. Whether the techniques employed in this article are extensible to other posets is uncertain.

These questions only scratch at the surface and, as is the case when one explores a new subject, each one begets many more. While this paper does not aim to solve (or even deeply examine) these problems, we hope the topics presented here serve as a intriguing jumping-off point. The interested reader is encouraged to examine the references for other results in this subject.

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