Dependent Random Choice

A Brief Introduction

A. Romer[1]

[1]Department of Mathematics
University of Victoria

Math 522

A Motivating Example

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$\boxed{\text{What class of graphs are triangle-free?}}$

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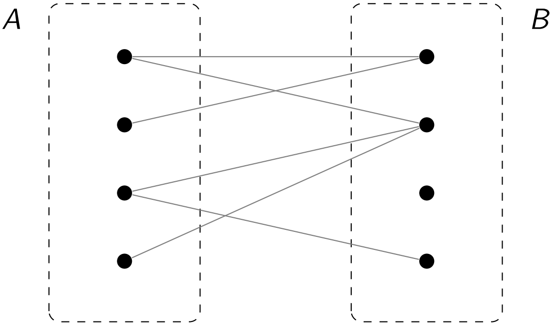

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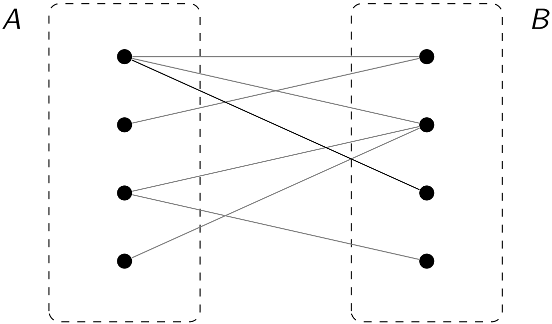

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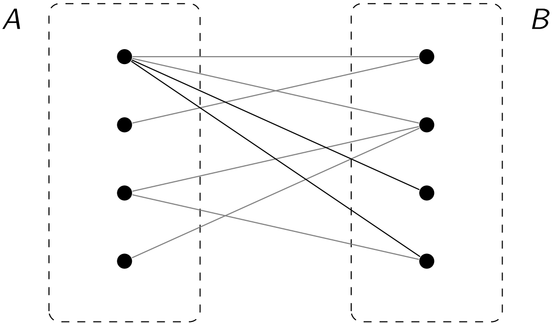

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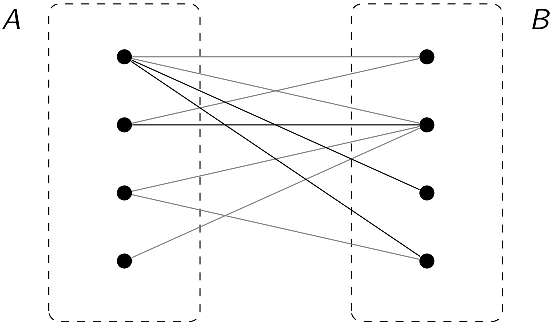

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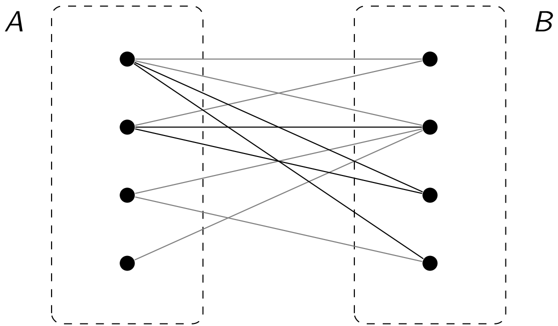

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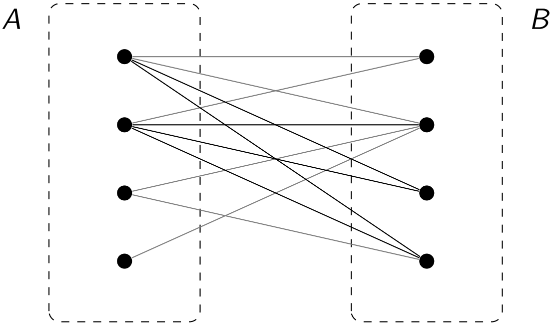

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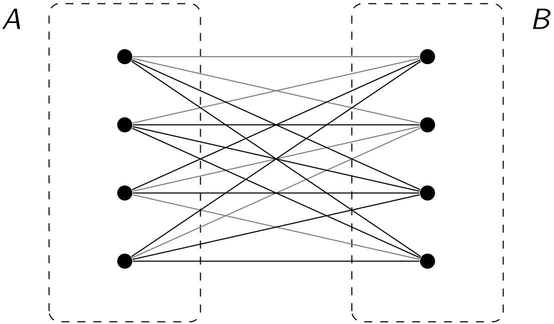

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(Perhaps entirely unsurprisingly), this sort of bipartite graph construction maximizes the number of edges in triangle free graphs on n vertices. While we won't prove this, hopefully it is at least intuitively clear why such a statement is reasonable.

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In general, the maximum number of edges in a triangle-free graph on n vertices is:

$$\left\lfloor \frac{n}{2} \right\rfloor \times \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

A Motivating Example

We will refer to the maximum number of edges in an n-vertex graph that does not contain a copy of some graph H as the *extremal number* of H, and we will write:

$$\mathrm{ex}(n, H).$$

On the previous slide, we showed that:

$$\mathrm{ex}(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

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It is natural to wonder what the extremal numbers of other graph classes are. In fact, this is one of the main topics in the field of extremal combinatorics. Perhaps the most obvious question is:

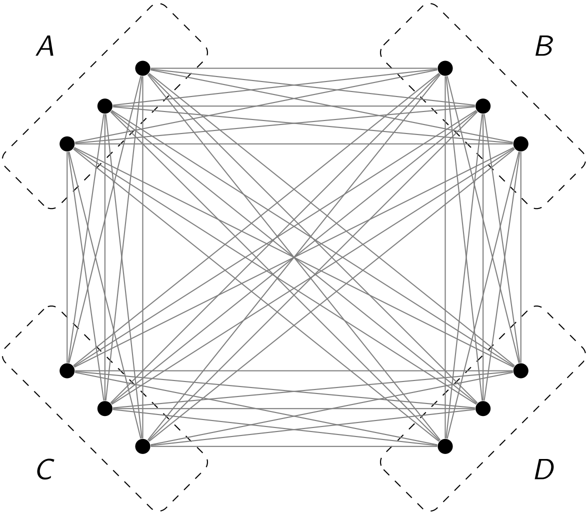
What is $\mathrm{ex}(n, K_r)$?

A Motivating Example

A natural place to start is with the following question:

$$\boxed{\text{What class of graphs are } K_r\text{-free?}}$$

A Motivating Example



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In much the same way that the bipartite graph is triangle-free, we can see that the r-partite graph is K_r-free. Specifically, in the previous example, we saw that the 4-partite graph does not contain any K_5's.

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In much the same way that the bipartite graph is triangle-free, we can see that the r-partite graph is K_r-free. Specifically, in the previous example, we saw that the 4-partite graph does not contain any K_5's.

It is therefore natural to assume that such a construction maximizes the number of edges in a K_r-free graph. Again, this is TRUE, although we will not prove it.

A Motivating Example

TO SUMMARIZE:

Bipartite graphs are good for making edge-y triangle-free graphs.

r-partite graphs are good for making edge-y graphs that do not contain cliques.

Something a bit more challenging...

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What if H is a bipartite graph? What is $\text{ex}(n, H)$?

What types of graphs are edge-y and also bipartite-free?

This problem is NOT EASY.

If a graph is edge-y, are there specific regions of the graph that are even edge-y-er?

Dependent random choice

The lemma of *dependent random choice* tells us that an “edge-y” graph does have “edgier” regions. Formally, it says:

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Lemma

Let a, d, m, n, r be positive integers. Let $G = (V, E)$ be a graph with $|V| = n$ vertices and average degree $d = 2|E(G)|/n$. If there is a positive integer t such that

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq a,$$

then G contains a subset U of at least a vertices such that every r vertices in U have at least m common neighbors.

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The statement of this lemma is a bit of a nightmare. We will look at it visually.

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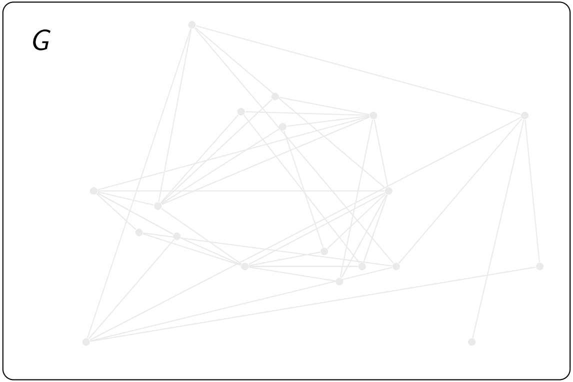

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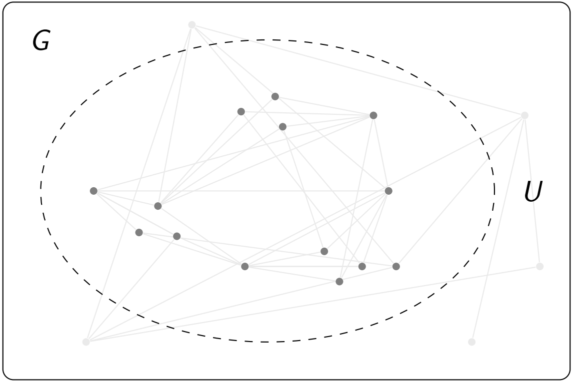

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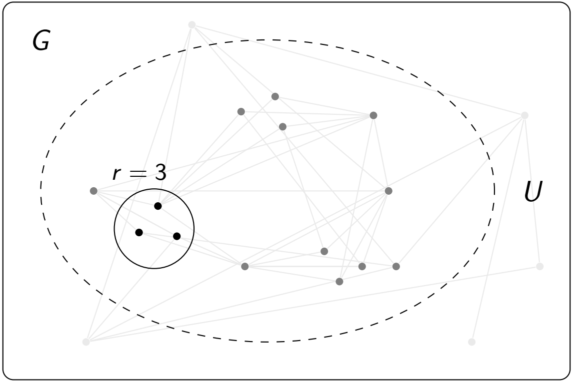

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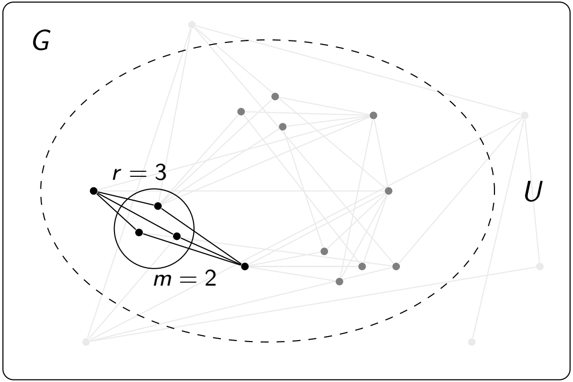

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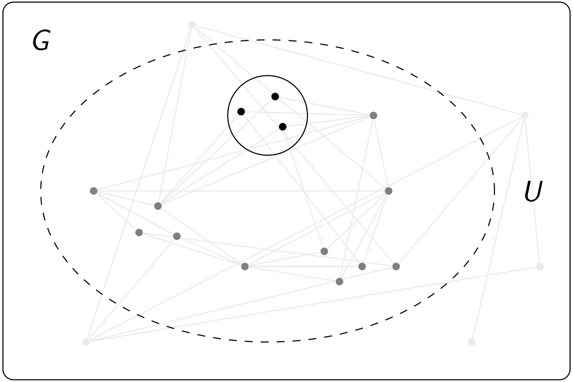

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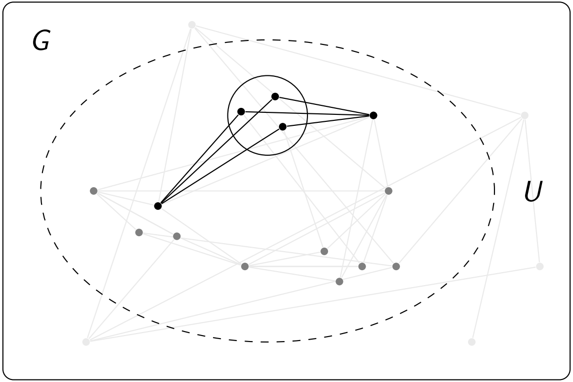

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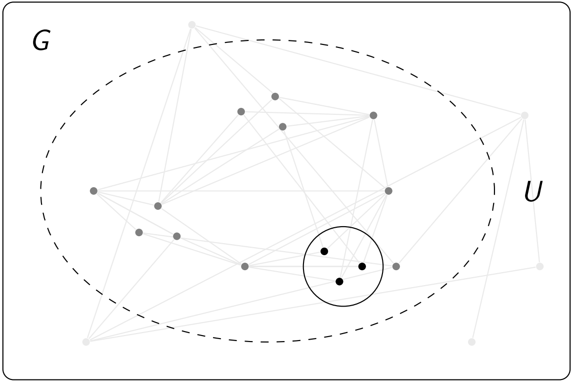

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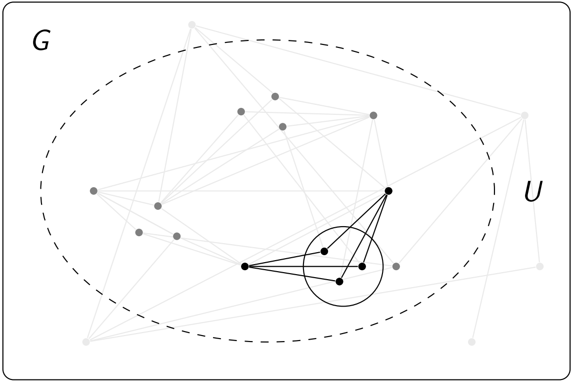

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So: dependent random choice tells us that we can always find a subset of vertices U such that

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Let's take a moment to understand the structure of this set U.

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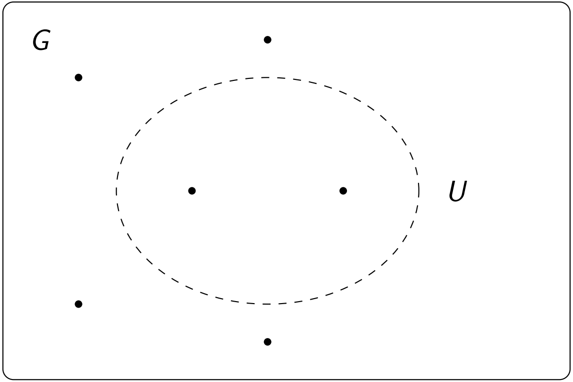
Suppose we have some graph where $|U| = 2$, $r = 2$ and $m = 4$.



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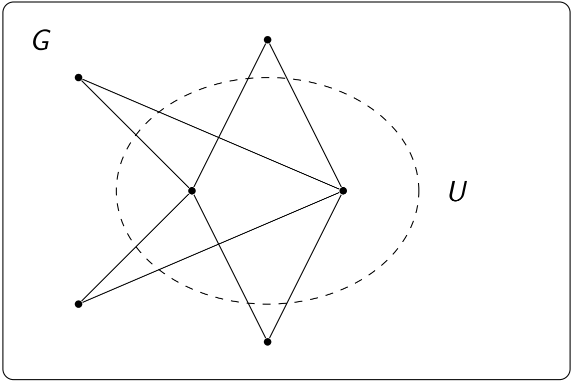

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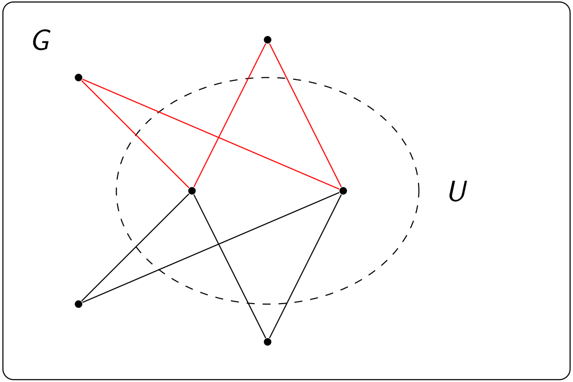

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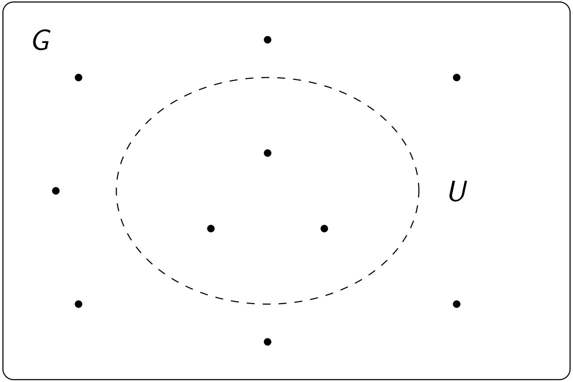
How about $|U| = 3$, $r = 2$ and $m = 5$?



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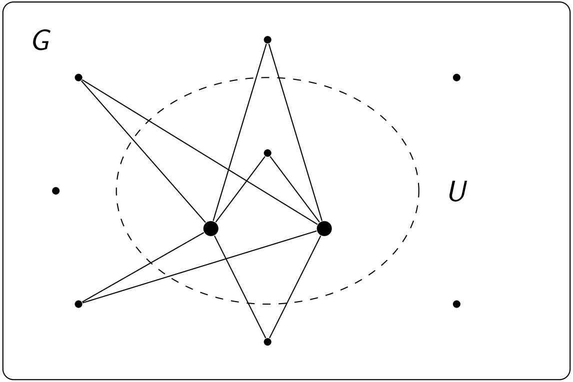

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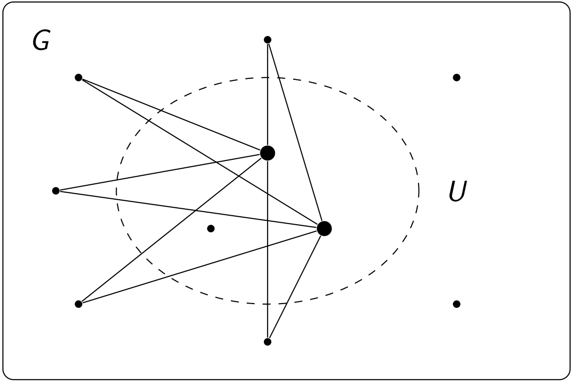

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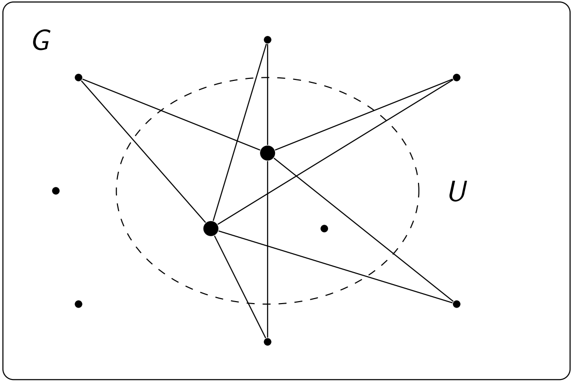

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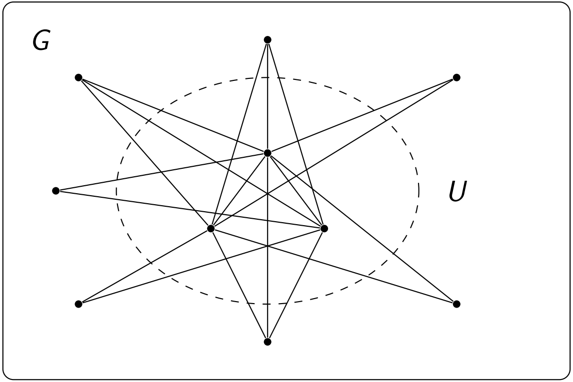

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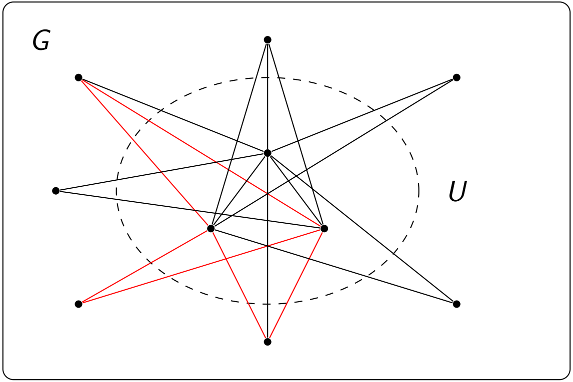

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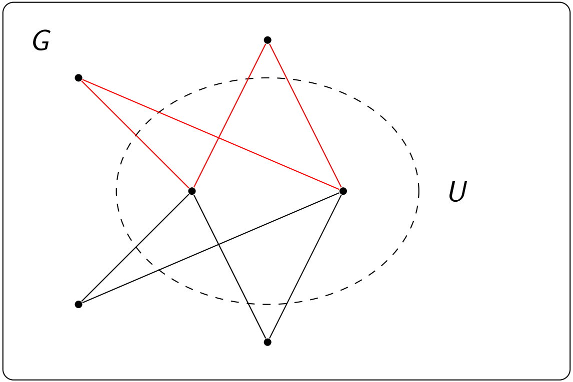

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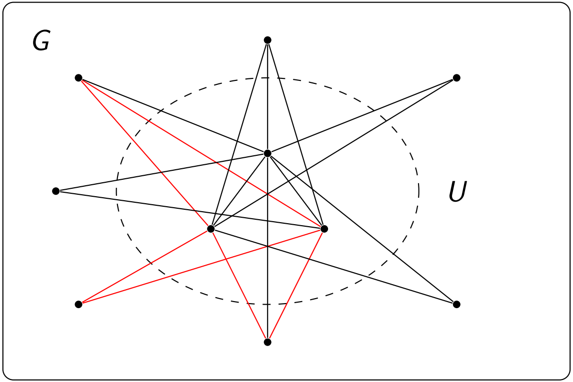

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In general, it seems that if a graph G has $|U| = |B|$, $r = \Delta(H_B)$, $m = |A| + |B|$, then it must contain a copy of $K_{|A|,|B|}$. Can this helps us determine an upper bound on $\mathrm{ex}(n, H)$?

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$$\boxed{\text{YES!}}$$

Dependent random choice

Theorem

Let H be a bipartite graph with parts A and B. If all vertices in B have degree at most $\Delta(H_B) = s$, then $ex(n, H) \leq cn^{2-\frac{1}{s}}$, where c is a constant whose value depends only on H.

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Proof idea/by example: We reverse-engineer

$$|U| \leq \frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t$$

to determine how many edges G must have in order for U to contain a copy of H, where $|U| = |B|$, $r = \Delta(H_B) = s$, and $m = |A| + |B|$ (as we saw in the above examples).

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Letting $d = 2cn^{1-1/s}$, $r = t = \Delta(H_B) = s$, $m = |A| + |B|$, we have:

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$$\begin{aligned} (2c)^s - \binom{n}{s}\left(\frac{|A|+|B|}{n}\right)^s &\geq (2c)^s - \left(\frac{en}{s}\right)^s\left(\frac{|A|+|B|}{n}\right)^s \\ &\geq (2c)^s - \left(\frac{e(|A|+|B|)}{s}\right)^s. \end{aligned}$$

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Finally, since c is simply some function of H, we can do some clever substitutions to get:

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Therefore, we have:

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This tells us that a graph G with average degree $2cn^{1-1/s}$ must contain a subset U with $|U| = |B|$ such that every set of $r = s = \Delta(H_B)$ vertices in U has $m = |A| + |B|$ neighbors.

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- If G has average degree $2cn^{1-1/s}$, then $|E(G)| \geq cn^{2-1/s}$ by the Handshaking Lemma.

CONCLUSION: If G has at least $cn^{2-1/s}$ edges, then G contains a bipartite graph. Therefore,

$$\mathrm{ex}(n, H) \leq cn^{2-\frac{1}{s}}.$$

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- Dependent random choice is a technique that allows us to make quantitative statements about small regions of a graph G, based only on the size of G and its average degree.
- This is useful because it provides an easy way to guarantee the presence of some (relatively) small graph H inside of G.

THANKS