Tight Bounds on 3-Neighbor Bootstrap Percolation

Master's Thesis

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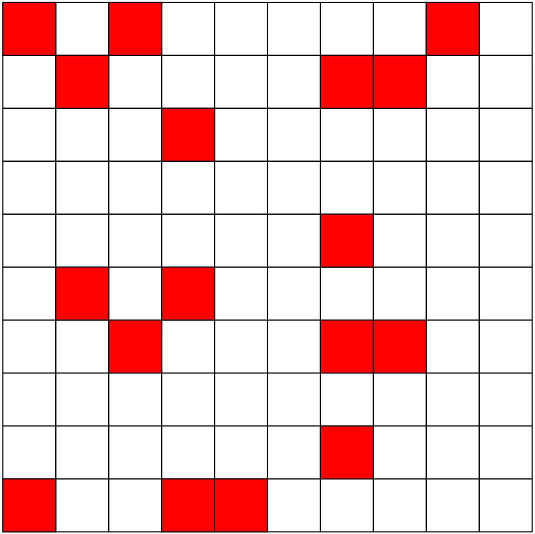
August 29, 2022

Bootstrap Percolation



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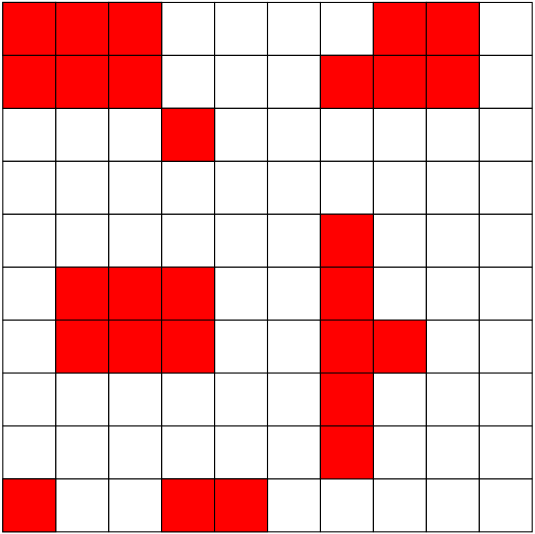

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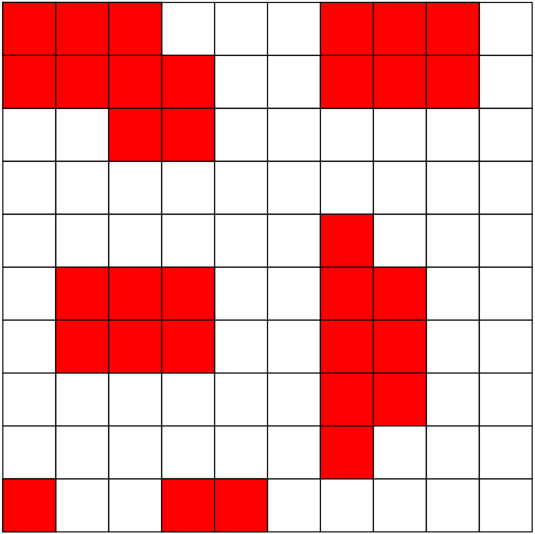

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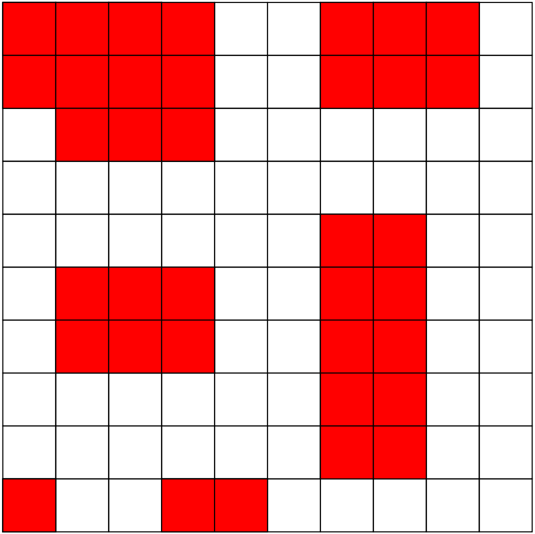

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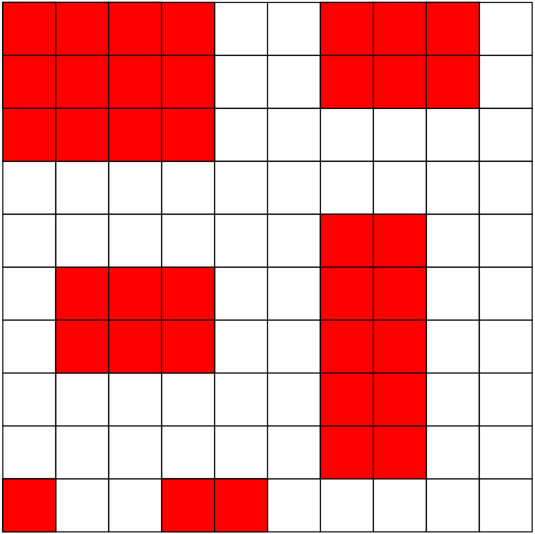


Definition

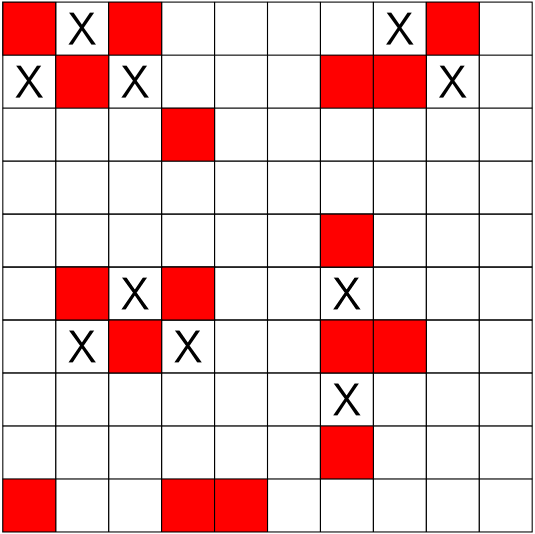
Let G be a graph, let v be a vertex of G, and let $A_t \subseteq V(G)$ be a set of infected vertices of G. We say that v becomes infected under r-neighbor *bootstrap percolation* if $|N_G(v) \cap A_t| \geq r$.

Explanation

If a cell is adjacent to at least r infected cells, it becomes infected.

Bootstrap Percolation

$r = 2$



Definition

A set of initially infected vertices in a graph G is said to be *lethal* or *percolate* if the infection eventually spreads to every vertex in G.

Definition

We denote by $m(G, r)$ the minimum size of a lethal set in G.

Definition

Let $\prod_{i=1}^{d}[a_i]$ represent the d-dimensional grid graph. For ease of notation, we define

$$m\left(\prod_{i=1}^{d}[a_i], r\right) = m(a_1, \ldots, a_d, r).$$

Explanation

The expression $m(a_1, \ldots, a_d, r)$ refers to the smallest lethal set on the d-dimensional grid graph with side lengths $a_1, \ldots, a_d$.

Bootstrap Percolation

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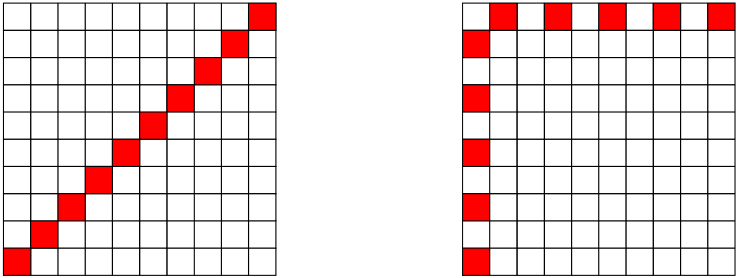
Question

What is $m(n, n, 2)$?

$m(n, n, 2)$

In the previous example, we observed that

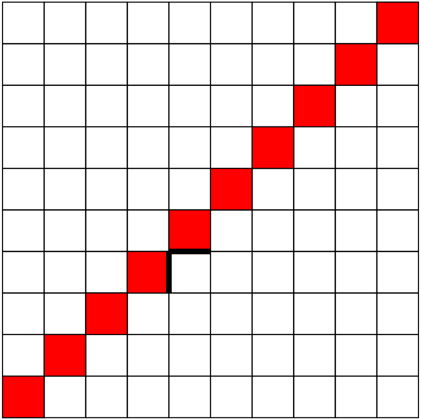
$$m(10, 10, 2) \leq 10 :$$



Is this the best we can do?

$m(n, n, 2)$

The perimeter of infection can never increase.



perimeter of infection $= 40$

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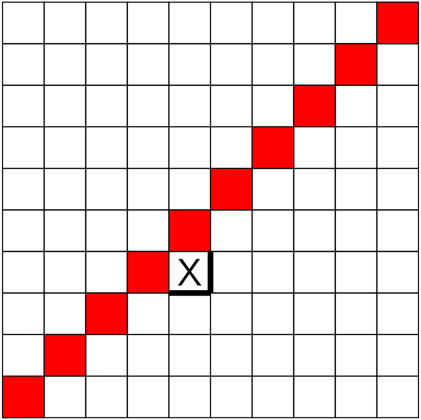

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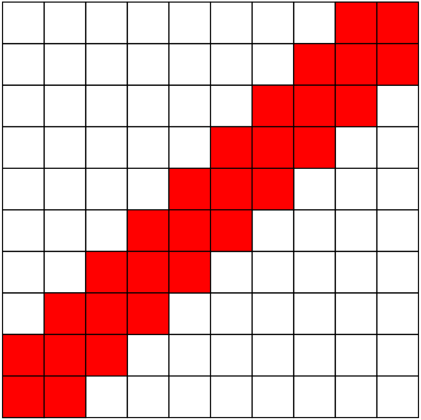

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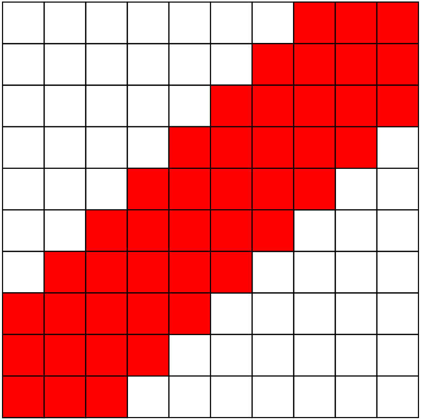

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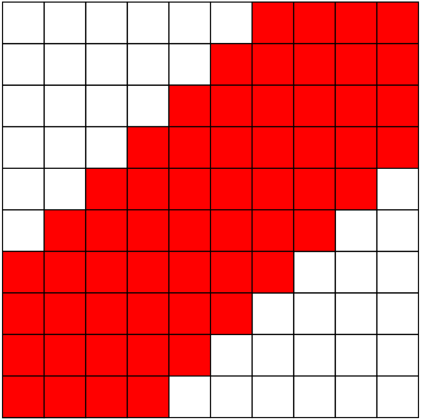

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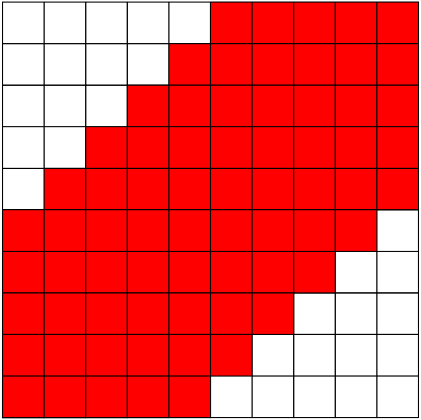

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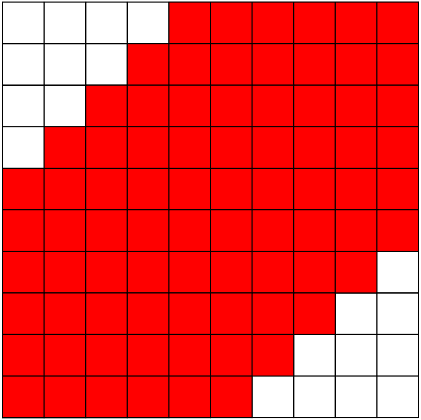

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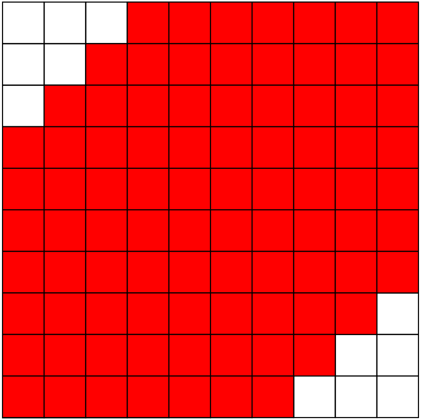

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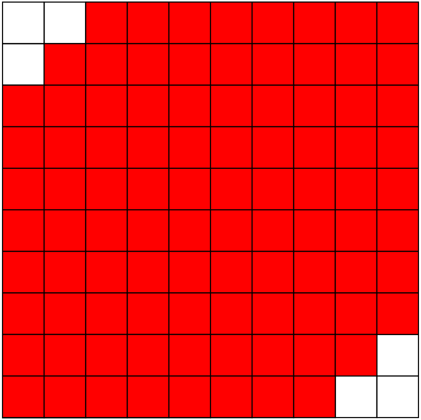

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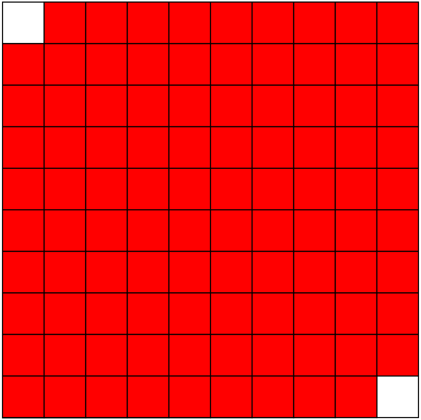

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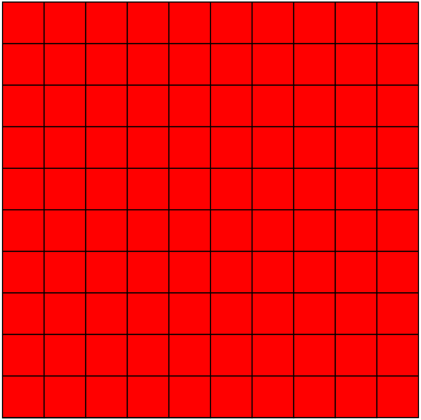

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The perimeter of the lethal set must be *at least* as large as the perimeter of the grid. For some lethal set A_0 in $[10] \times [10]$, we therefore have:

$$\text{perimeter}(A_0) \geq \text{perimeter}([10] \times [10]) = 4(10).$$

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Note that the perimeter of A_0 is *at most* $4|A_0|$, and so

$$4|A_0| \geq 40 \implies |A_0| \geq 10.$$

We have seen an example where $|A_0| = 10$, so we conclude that

$$m(10, 10, 2) = 10.$$

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It should (hopefully) not be too surprising that this argument generalizes to

$$m(n, n, 2) = n.$$

Summary

Furthermore, the same idea can be applied to rectangular grids to obtain:

$$m(a_1, a_2, 2) \geq \left\lceil \frac{1}{2}(a_1 + a_2) \right\rceil.$$

Let us now turn to the results of this research:

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Tight Bounds on 3-Neighbor Bootstrap Percolation

- Tight Bounds: exact size of smallest lethal set
- on 3-Neighbor: with infection threshold of 3
- Bootstrap Percolation: (on 3D rectangular grids)

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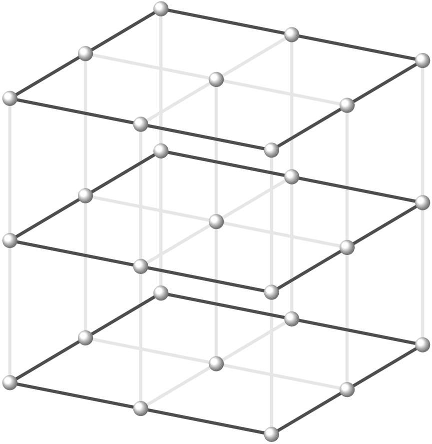
We would like to determine the exact value of $m(a_1, a_2, a_3, 3)$.

What is the size of the smallest lethal set on all 2- and 3-dimensional grids under 3-neighbor bootstrap percolation?

Example: $[3] \times [3] \times [3]$



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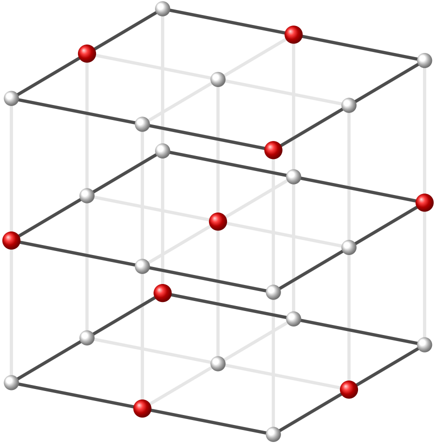

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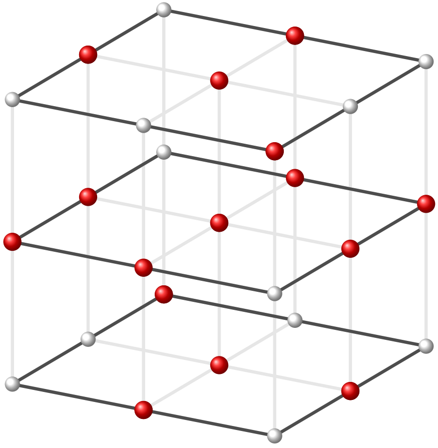

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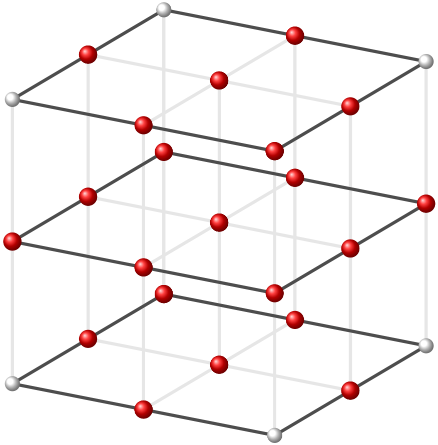

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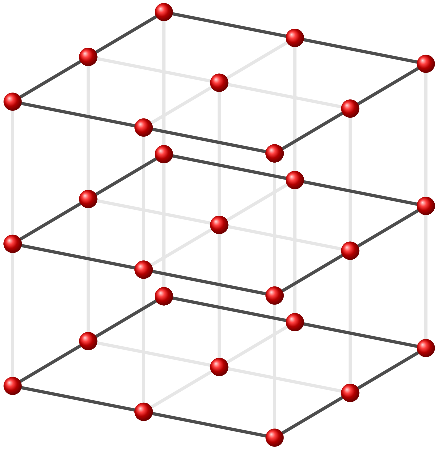

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Results

1. For all $a_1, a_2, a_3 \geq 11$, we have that

$$m(a_1, a_2, a_3, 3) = \left\lceil \frac{1}{3}(a_1 a_2 + a_2 a_3 + a_3 a_1) \right\rceil;$$

We know the smallest lethal set on all sufficiently large grids.

2. For $G = C_{a_1+1} \square C_{a_2+1} \square C_{a_3+1}$ and $a_1, a_2, a_3 \geq 11$,

$$m(a_1, a_2, a_3, 3) + 1 \leq m(G, 3) \leq m(a_1, a_2, a_3, 3) + 2;$$

We know (within 1) the smallest lethal set on all sufficiently large tori.*

3. $m(n, n, 3) = \frac{1}{3}(n^2 + 2n)$ if and only if $n = 2^k - 1$, for some $k > 0$.
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Strategy

How do we obtain this result?

Our basic approach will be:

1. Determine a lower bound on $m(a_1, a_2, a_3, 3)$;
2. Find lethal sets that match this lower bound.

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1. *Determine a lower bound on* $m(a_1, a_2, a_3, 3)$*;*
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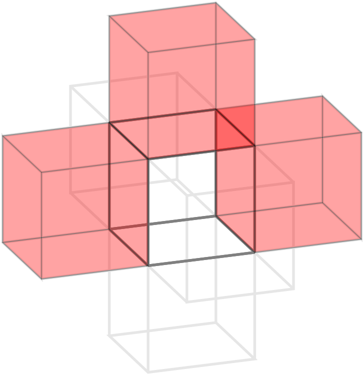
For a lower bound, *we can generalize the perimeter argument to 3-neighbor percolation in three dimensions.*

Lower Bound



When a cube becomes infected, the total surface area of infection cannot increase.

Lower Bound

The surface area of the lethal set must be *at least* as large as the surface area of the grid. For some lethal set A_0 in $G = [a_1] \times [a_2] \times [a_3]$, we therefore have:

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Note that the surface area of A_0 is *at most* $6|A_0|$, and so

$$6|A_0| \geq 2(a_1 a_2 + a_2 a_3 + a_3 a_1) \implies |A_0| \geq \left\lceil \frac{a_1 a_2 + a_2 a_3 + a_3 a_1}{3} \right\rceil.$$

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This gives us the following lower bound:

$$m(a_1, a_2, a_3, 3) \geq \left\lceil \frac{a_1 a_2 + a_2 a_3 + a_3 a_1}{3} \right\rceil.$$

We are shooting for $m(a_1, a_2, a_3, 3) \geq \lceil (a_1 a_2 + a_2 a_3 + a_3 a_1)/3 \rceil$.

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1. ~~Determine a lower bound on $m(a_1, a_2, a_3, 3)$~~;
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   2.1 Find a good set of “atomic” examples;
   2.2 Assemble these examples into all larger “molecular” grids.

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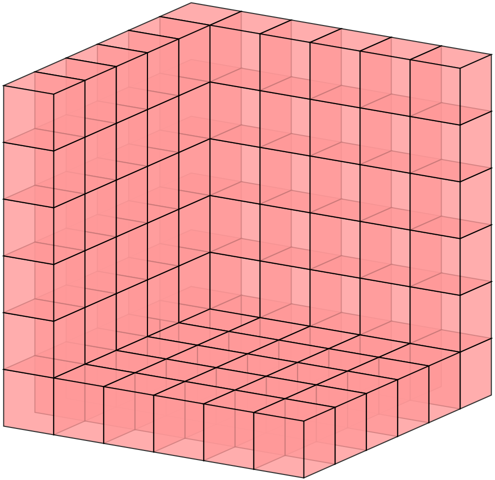
We present our methodology for generating these "atomic" pieces.

Three Walls

Lethal infections on 3 mutually perpendicular faces of $G = [a_1] \times [a_2] \times [a_3]$ are lethal on G.



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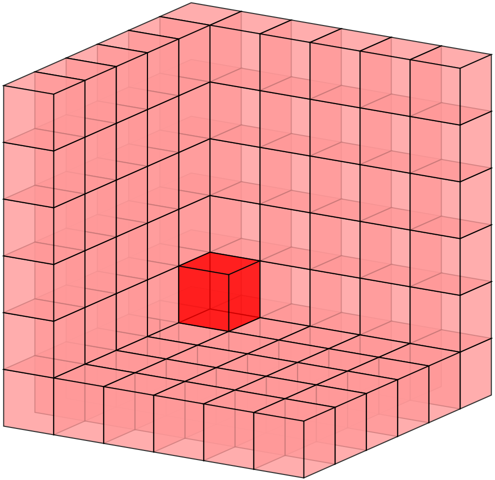

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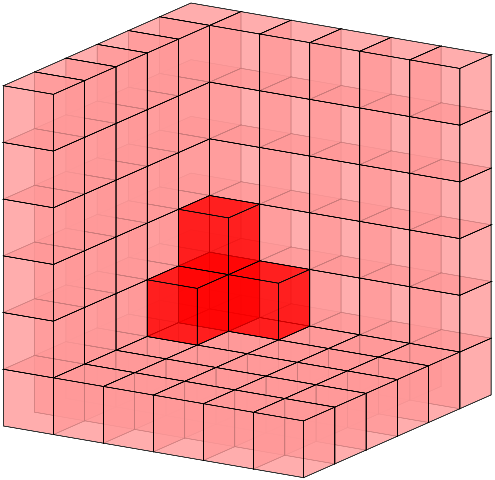

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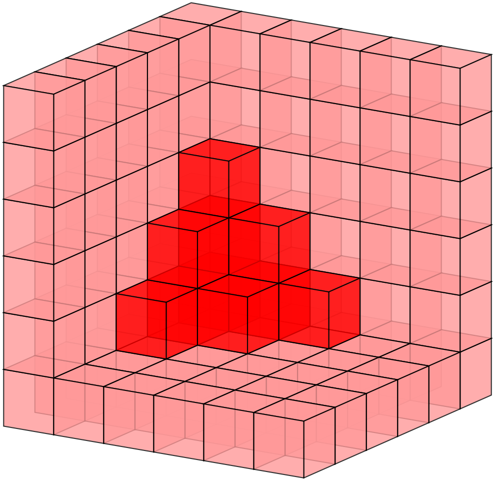

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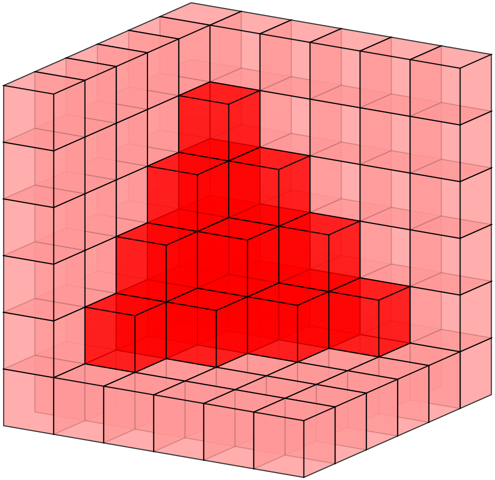

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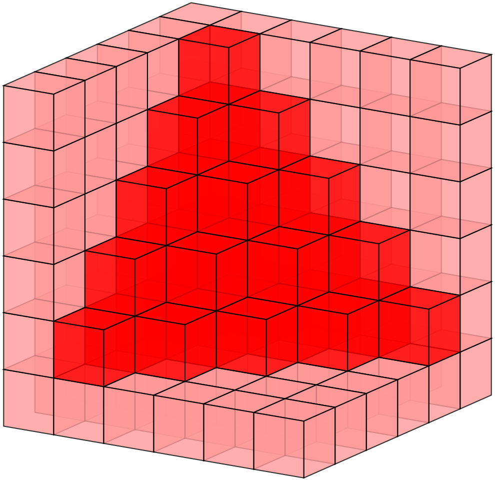

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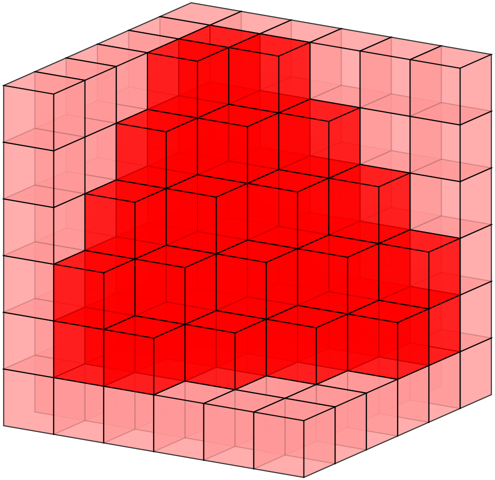

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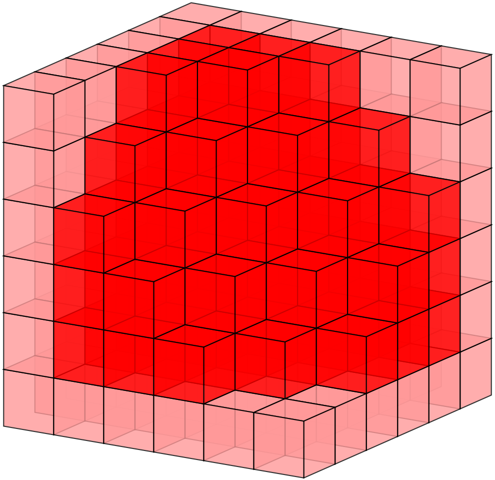

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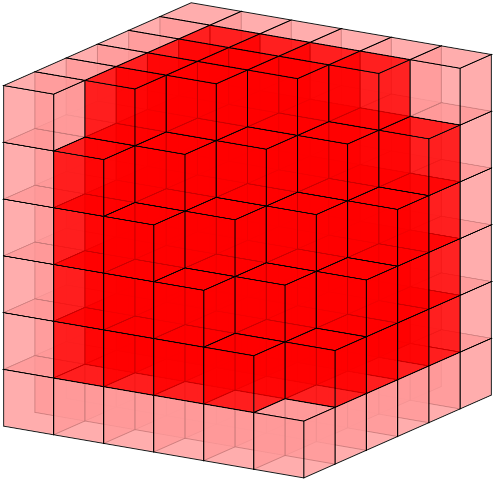

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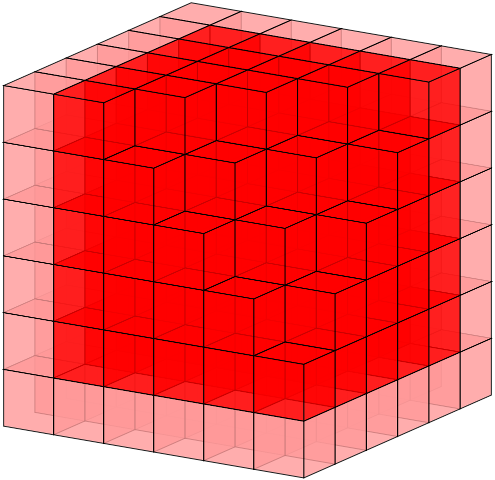

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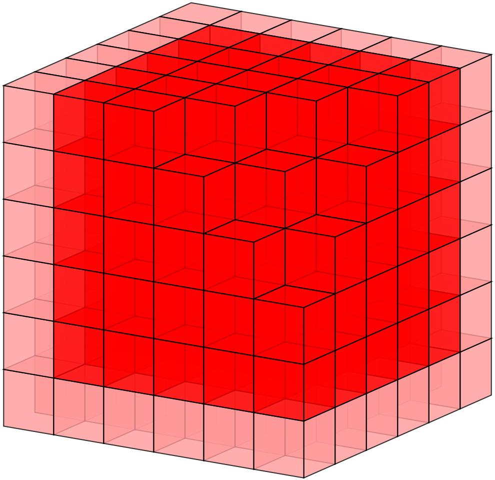

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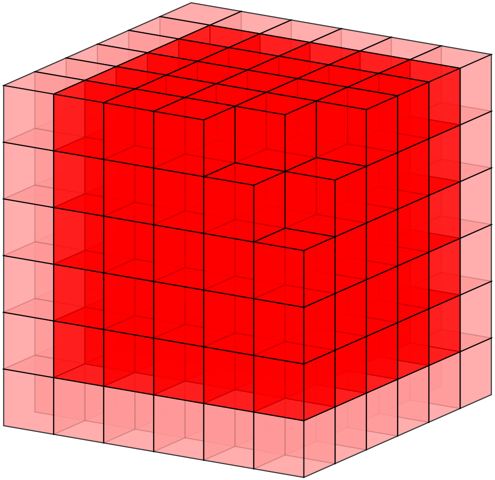

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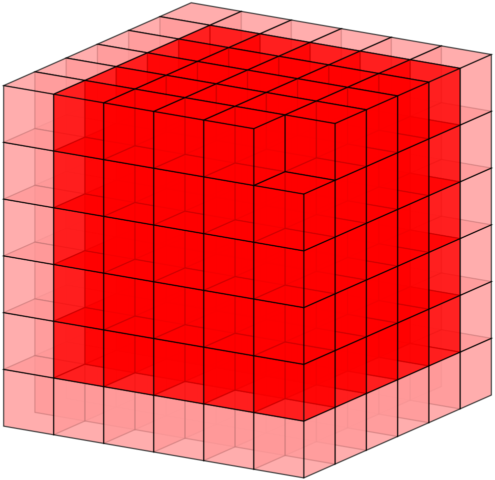

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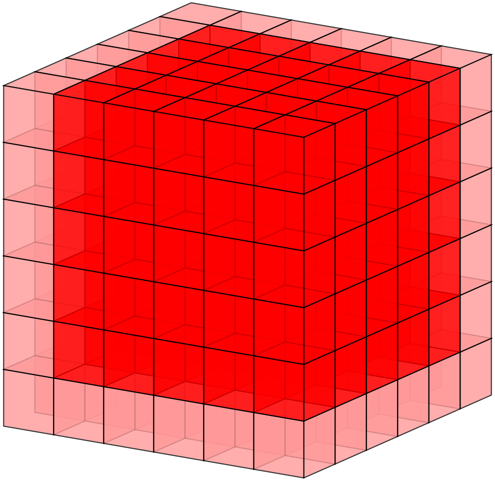
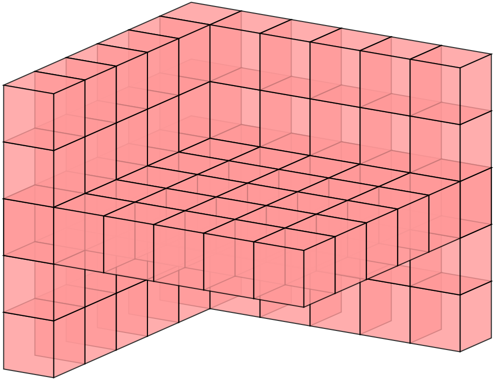


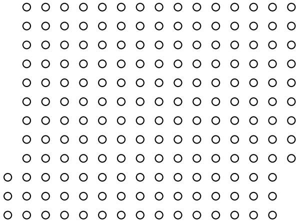
Three Walls

This is true for lethal infections on *any* 3 mutually perpendicular walls.



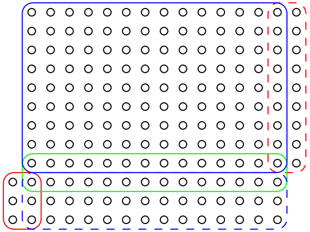
Origami

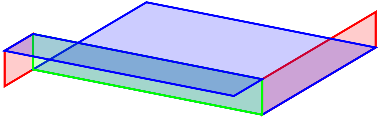
Suppose we have a 2D grid.



Origami

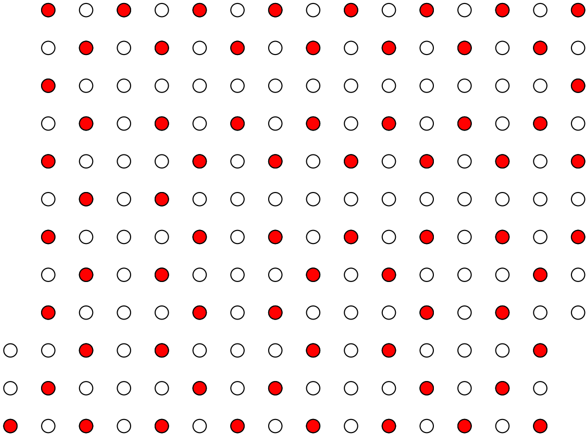
Suppose we have a 2D grid. We can imagine folding this flat grid up into a 3D structure.





Origami

By our "three walls" argument, any lethal set on this 2D grid will also be lethal on the resulting 3D structure.



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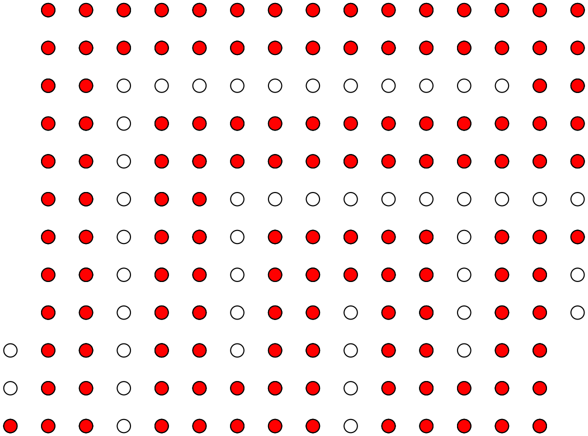

Origami

Applying this process to a variety of 2D grids, we are able to obtain tight bounds on:

$$m(a_1, a_2, 2)^1 : \begin{cases} a_1, a_2 \equiv 0 \bmod 3, \text{ and } a_1 \not\equiv a_2 \bmod 6; \\ a_1, a_2 \equiv 2 \bmod 3, \text{ and } a_1 \not\equiv a_2 \bmod 6; \\ a_1 \equiv 0 \bmod 3, \text{ and } a_2 = 3. \end{cases}$$

$$m(a_1, a_2, 3)^1 : \begin{cases} a_1 \equiv 3 \bmod 6, \text{ and } a_2 \equiv 1 \bmod 2; \\ a_1 \geq 2, \text{ and } a_2 \in \{3, 6\}; \\ a_1 \equiv 3 \bmod 6, \text{ and } a_2 = 4. \end{cases}$$

We can get a lot of tight constructions for grids of the form $[a_1] \times [a_2] \times 2$ *and* $[a_1] \times [a_2] \times 3$*.*

[1] Some small examples are omitted for the purposes of clarity.

Strategy

Our basic approach will be:

1. ~~Determine a lower bound on $m(a_1, a_2, a_3, 3)$~~;
2. *Find lethal sets that match this lower bound.*
    2.1 ~~Find a good set of "atomic" examples~~;
    2.2 Assemble these examples into all larger "molecular" grids.

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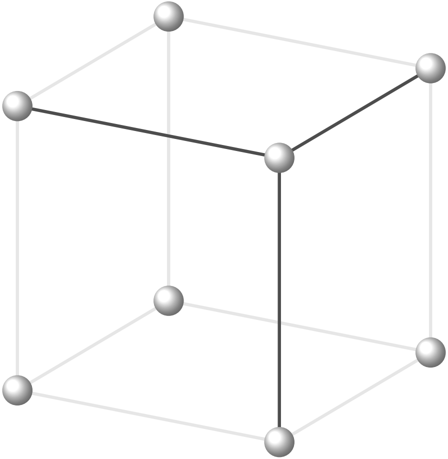
We obtain tight lethal constructions on grids of height 5, 6, and 7, and then use these constructions to obtain tight lethal constructions on *all grids*.

Recursion



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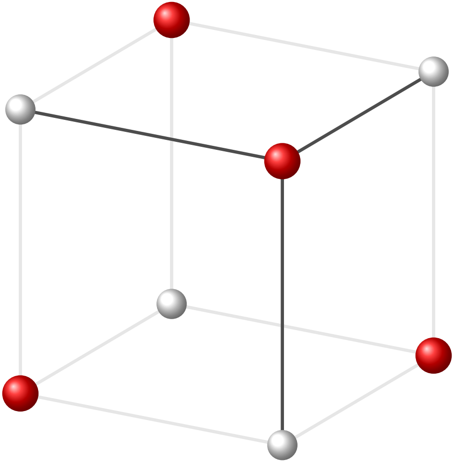

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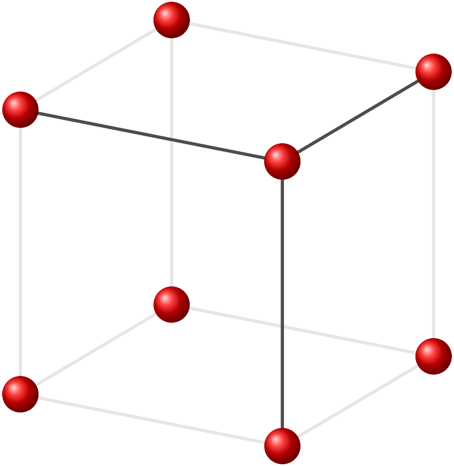
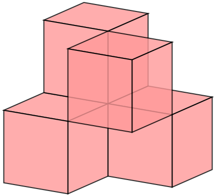


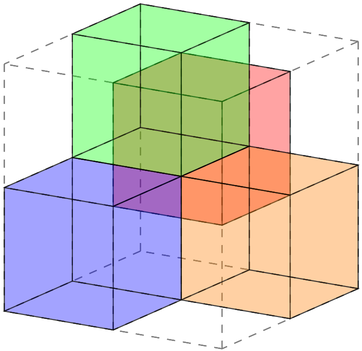
Recursion

What if we replace each infected node with a minimum percolating set?



1. Will it percolate? YES
2. What is the size of this new set S?

Recursion



$$a_1 \times b_1 \times c_1 \implies (a_1 b_1 + b_1 c_1 + c_1 a_1)/3$$
$$a_2 \times b_2 \times c_1 \implies (a_2 b_2 + b_2 c_1 + c_1 a_2)/3$$
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$$a_1 \times b_2 \times c_2 \implies (a_1 b_2 + b_2 c_2 + c_2 a_1)/3$$

Recursion

What is the size of this new set S?

We can see that

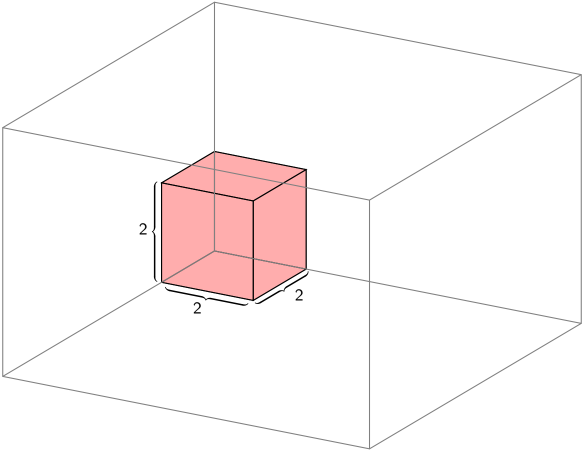
$$|S| = (a_1 b_1 + b_1 c_1 + c_1 a_1)/3 + (a_2 b_2 + b_2 c_1 + c_1 a_2)/3 \\ + (a_2 b_1 + b_1 c_2 + c_2 a_2)/3 + (a_1 b_2 + b_2 c_2 + c_2 a_1)/3$$

which we can simplify to

$$\frac{(a_1 + a_2)(b_1 + b_2) + (b_1 + b_2)(c_1 + c_2) + (c_1 + c_2)(a_1 + a_2)}{3}.$$

This is the minimum size of a percolating set on our $(a_1 + a_2) \times (b_1 + b_2) \times (c_1 + c_2)$ grid!

Example: $[8] \times [8] \times [5]$



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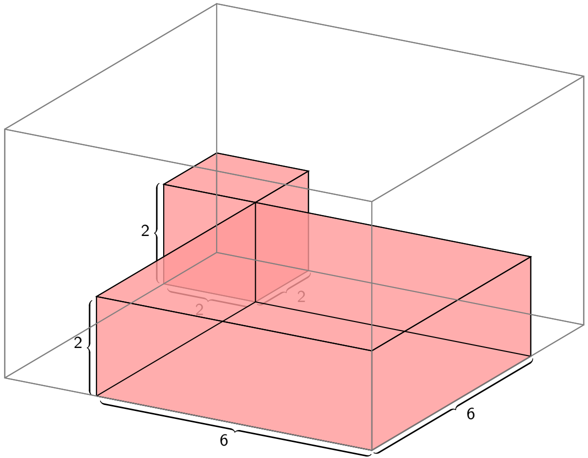

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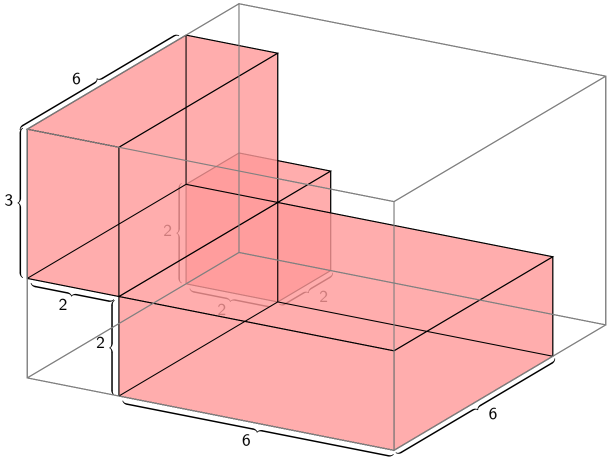

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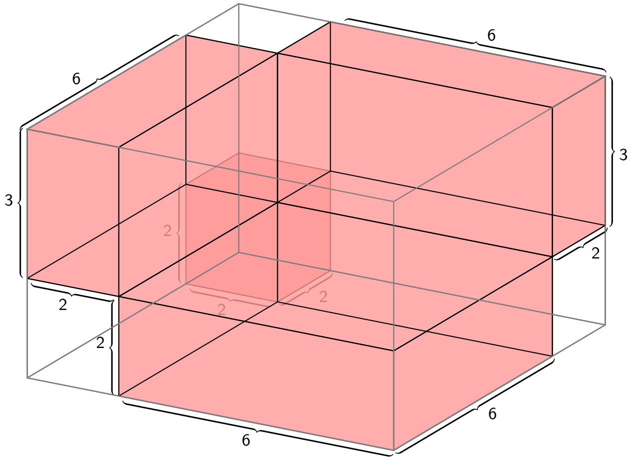

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We use these constructions to generate optimal lethal sets on *all* grids of size at least 11.

One Proof

Some Facts:

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The following diagram illustrates how to obtain a perfect lethal set on the grid $[3b_1 + r_1] \times [3b_2 + r_2] \times [3b_3 + r_3] = [a_1] \times [a_2] \times [a_3]$.

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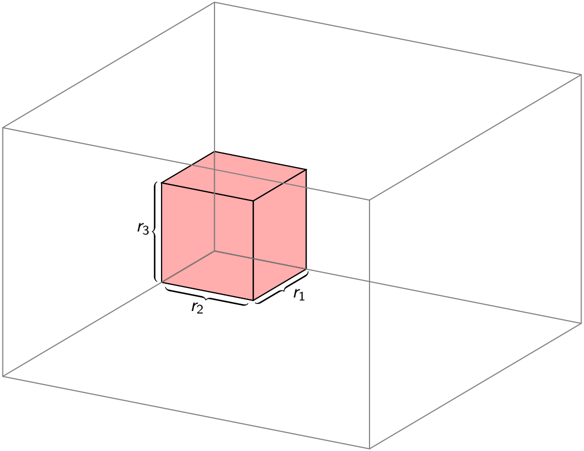

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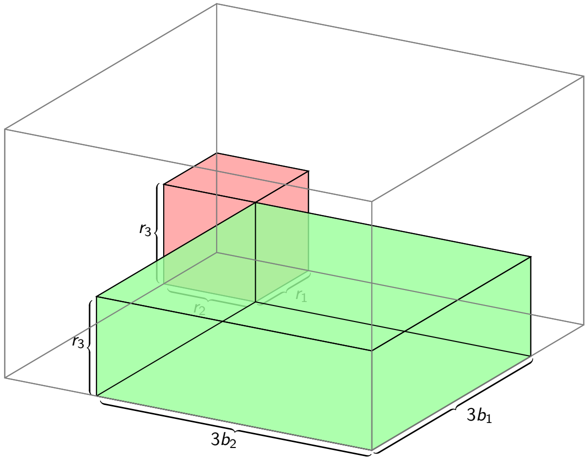

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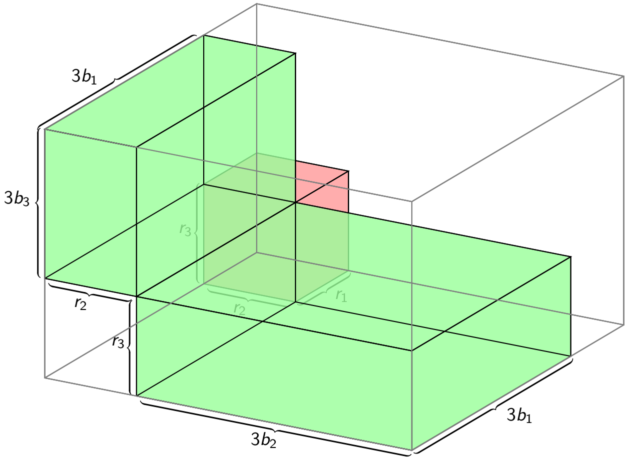

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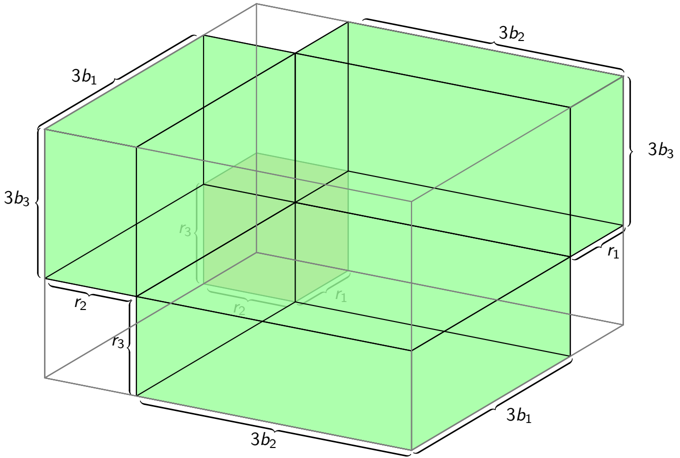

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5. We conclude that

$$m(a_1, a_2, a_3, 3) = \left\lceil \frac{a_1 a_2 + a_2 a_3 + a_3 a_1}{3} \right\rceil$$

for all for $a_1, a_2, a_3 \geq 11$.

THANKS

https://ahblay.pythonanywhere.com