Dependent Random Choice

Abel Romer

April 11, 2021

1 Introduction

A large number of graph-theoretic problems center around the issue of embedding a small graph H inside of a larger graph G. While questions of embeddings can sometimes be answered by simply looking at structural properties of the graphs (e.g. are they bipartite, complete, cycles, etc.), it is often the case that more robust or generalizable strategies are needed. One such strategy, which has grown in prominence and success in recent years, is that of *dependent random choice*. In the simplest of terms, dependent random choice is a probabilistic tool that can be used to guarantee the existence of highly connected areas in all graphs with "enough" edges.

Naturally, such a description gives rise to a number of questions. These include: What does probability mean in the context of graphs? What do we mean by "enough edges"? How connected is "highly connected"? How do we find such an area? Additionally, one might wonder about the specific problems that employ this technique: When is it useful to find a set of highly connected vertices? What does that help us prove? Can we apply this strategy to all types of graphs? How extensible is it?

In this paper, we address the above questions, in roughly the order that they are presented. We first examine the use of probability in graph theory, and present some simple examples that illustrate its value. We then move on to the specific technique of dependent random choice, where we flesh out the description presented above. We present both technical and intuitive explanations of the lemma, along with a (relatively) simple application. Finally, we explore some problems, the solutions of which rely on the application of dependent random choice.

1.1 The Probabilistic Method

We typically think of problem solving as the process of determining answers. In essence, we are given a question and expected to provide a (somewhat) concrete result. When applying the probabilistic method, however, we simply *show* that there is some positive probability that a solution exists (within some broader probability space). This then implies that there is some circumstance where we achieve the desired outcome. It is worth noting that this method is *non-constructive*; while it proves that something exists, it tells us very little about how to find it.

The probabilistic method has been used extensively in graph theory and combinatorics at large to prove the existence of certain structures. Perhaps the most classical application of the technique is exhibited in a proof by Erdős regarding the Ramsey numbers. [The following proof is given by Balachandran in their paper on the probabilistic method [1].]

Proposition 1.1. *For $k \geq 3$, $R(k,k) > \lfloor 2^{k/2} \rfloor$.*

Proof: Let $G = K_n$ be a complete graph on $n = \lfloor 2^{k/2} \rfloor$ vertices. We show that, given a uniformly random coloring of the edges of G, the probability that there exists a monochromatic induced subgraph on a subset of k vertices of G is less than 1. This implies that there exists at least one edge-coloring of G where no k-vertex induced subgraph is monochromatic, and therefore $R(k,k) > \lfloor 2^{k/2} \rfloor$.

Two-color the edges of G uniformly at random. Let $R \subset V(G)$ be a set of k vertices, and A_R be the event that the edges of the subgraph induced by R are monochromatic. We then have that the probability of A_R is:

$$\mathbb{P}(A_R) = 2 \cdot \frac{2^{\binom{n}{2}}/2^{\binom{k}{2}}}{2^{\binom{n}{2}}} = \frac{2}{2^{\binom{k}{2}}} = 2^{1-\binom{k}{2}}.$$

Summing over all possible events A_R provides an upper bound on the probability that G contains a monochromatic induced subgraph on k vertices. As there are $\binom{n}{k}$ possible subsets R, we have:

$$\sum_{R \subset V(G), |R|=k} \mathbb{P}(A_R) = \binom{n}{k} \mathbb{P}(A_R) = \binom{n}{k} 2^{1-\binom{k}{2}}.$$

We now show that the above expression is less than 1, thus implying that there exists at least one coloring of the edges of G that does not contain any monochromatic induced subgraphs on k vertices. Recalling that $\binom{n}{k} \leq \frac{n^k}{k!}$, we have for $k \geq 3$:

$$\binom{n}{k} 2^{1-\binom{k}{2}} < \frac{n^k}{k!} \cdot 2^{1-(k^2/2)+(k/2)} = \frac{2^{1+(k/2)}}{k!} \cdot \frac{n^k}{2^{k^2/2}}.$$

Finally, plugging in $n = \lfloor 2^{k/2} \rfloor$:

$$\frac{2^{1+(k/2)}}{k!} \cdot \frac{n^k}{2^{k^2/2}} = \frac{2^{1+(k/2)}}{k!} \cdot \frac{2^{k^2/2}}{2^{k^2/2}} = \frac{2^{1+(k/2)}}{k!} < 1.$$

Therefore, we conclude that there must exist at least one coloring of the edges of G that does not contain any monochromatic induced subgraphs on k vertices. This implies that $R(k,k) > \lfloor 2^{k/2} \rfloor$. $\square$

The above example helps illustrate the power of the probabilistic method. Rather than searching through all possible colorings to find a particular example that satisfies the constraints of the problem, we simply show that such an example must exist. In the circumstance where it is sufficient to simply know of the existence of such an example, this process can be a significant time-saver, as it provides a (somewhat) prescriptive approach, rather than requiring an outright search.

Dependent random choice provides a slightly more complicated example of the application of probabilistic techniques to graph-theoretical concepts. However, the fundamental idea remains the same. We show that, given a graph with average degree d, there exists a subset U of a vertices (the size of which depends on d, among other variables), where all subsets of U have a lot of common neighbors. We do not show how to find this subset U; we simply guarantee that it exists probabilistically.

1.2 Applications

Before we dive into the formal statement of the lemma of dependent random choice, let us take a moment to familiarize ourselves with the types of problems that it can help us solve. As we suggested in the introduction, problems of embeddings are particularly well-suited to dependent random choice. The reason for this should be somewhat clear. Since dependent random choice tells us about this existence of certain edge-dense regions in a graph, it can be used to guarantee the presence of certain small graphs within these regions.

Perhaps the most accessible of such problems are those concerned with the *extremal number* of a graph H. These problems pose the following question: How many edges can we add to a graph on n vertices such that this graph does not contain a copy of H? We define the extremal number as follows:

Definition 1.1. Let G be a graph on n vertices. Let the *extremal number* $\text{ex}(n, H)$ of H be the maximum number of edges in G such that G does not contain any copies of H.

For example, we might wish to determine $\text{ex}(n, K_3)$; that is, how many edges can we add to an n-vertex graph without creating any triangles? Or we might wish to know the answer to this question for all complete graphs K_r. Or bipartite graphs. The list goes on. This problem has been extensively studied, and a number of results are known. To list a few:

- $\text{ex}(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$: This result is better known as Mantel's theorem, and states that the triangle-free graph with the largest number of edges is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

- $\mathrm{ex}(n, K_r)$: The extremal number of K_r is precisely the number of edges in a complete $(r-1)$-partite graph with partitions of roughly (within floor/ceiling) equal size. This is result is due to Turán, and is simply a generalization of Mantel's theorem.
- $\mathrm{ex}(n, H) \leq cn^{2-\frac{1}{r}}$: This result applies to bipartite graphs H with maximum degree r. Here we see that any n-vertex graph with more than some constant multiple c of $n^{2-\frac{1}{r}}$ edges must contain a copy of H.

While the first two results may seen somewhat intuitive, the third is less obvious. Where does the $n^{2-\frac{1}{r}}$ term come from? What is the structure of some graph G that does not contain bipartite H? We shall see in the following sections that this bound actually relies on the lemma of dependent random choice, and is therefore not constructive. We will simply show that in a graph with at least $cn^{2-\frac{1}{r}}$ edges, there must exist a region of high enough connectedness to ensure the presence of a bipartite H.

Finally, at the end of the paper, we demonstrate how dependent random choice can be used to determine an upper bound on Ramsey numbers of the cube, and discuss some further developments related to this problem.

2 Dependent Random Choice

We have already seen a high-level description of dependent random choice, and understand it to be a probabilistic process through which we are able to determine certain properties about specific small regions in a graph. Before we see the formal statement of this process, we shall present one final example, with the aim of further building intuition about the upcoming result. [The following example is borrowed from a lecture by Professor Yufei Zhao at MIT [5].]

> Suppose we wish to model acquaintances between students at a university. Let us suppose that every student knows some (fairly large) number d of other students. Dependent random choice says that we can always find a large subset U of the student body, such that every group of r students in U have at least a certain number (say m) of common friends (figure 1). We might think of this group U as the math department, and each group r as a particular math course. In other words, we can always "zoom in" on a particular group (region), where most people (vertices) know each other (are connected).

Figure 1 provides a representation of what some subsets of size r of this set U might look like. However, note that this figure *does not* include all of the edges in U. Because each subset of U of size r has at least m common neighbors, it is more accurate to think of U as something resembling a complete graph. An example of such a set U is given in figure 2.

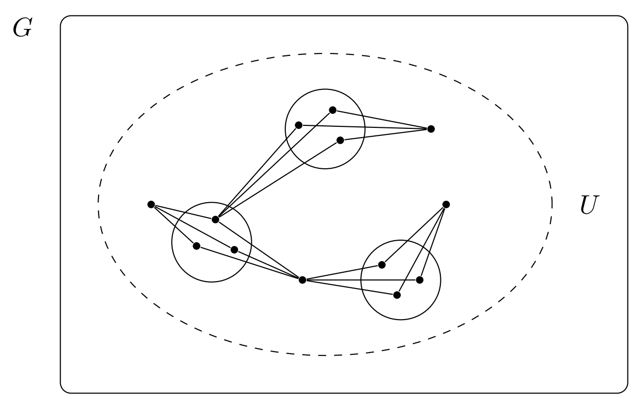

Figure 1: A visual explanation of dependent random choice ($r = 3$, $m = 2$). Not all edges are shown. Examples of groups of r vertices are circled and common neighbors are represented by edges. Note that we consider *all* groups of r vertices, not simply the ones shown in the figure.

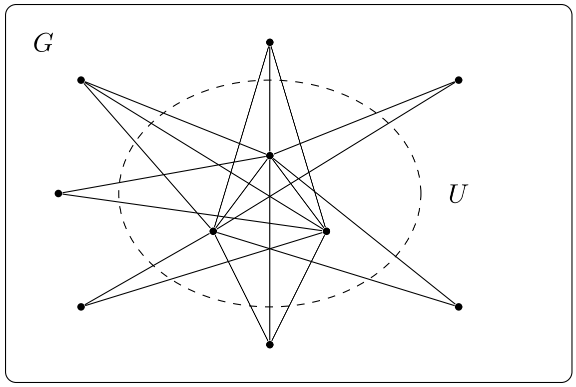

Figure 2: A visualization of the subset U. Here we have $|U| = 3$, $r = 2$ and $m = 5$.

With this example in mind, we present the lemma of dependent random choice. [This particular instance of the lemma is due to Fox and Sudakov [2], with proof ideas from [2] and [3].]

Lemma 2.1. *Let a, d, m, n, r be positive integers. Let $G = (V, E)$ be a graph with $|V| = n$ vertices and average degree $d = 2|E(G)|/n$. If there is a positive integer t such that*

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq a,$$

then G contains a subset U of at least a vertices such that every r vertices in U have at least m common neighbors.

Proof: We show that the expression on the LHS of the above inequality finds a subset $U \subseteq V(G)$ of at least a vertices that satisfies the constraints of the lemma. We consider the two terms of the LHS independently.

1. $\frac{d^t}{n^{t-1}}$: Let T be a set of t vertices of G chosen uniformly at random with repetition, let $A = N(T)$ be the common neighborhood of T, and let $X = |A|$. We show that d^t/n^{t-1} gives a lower bound on the expected size of A. In other words, we show that

$$\mathbb{E}[X] \geq \frac{d^t}{n^{t-1}}.$$

Recall that the expectation is simply the sum of the probabilities. Therefore:

$$\mathbb{E}[X] = \sum_{v \in V(G)} \mathbb{P}(v \in A).$$

Note that the probability that a vertex v is in A is precisely the same as the probability that v is adjacent to all the vertices in T. Therefore:

$$\mathbb{E}[X] = \sum_{v \in V(G)} \mathbb{P}(T \subseteq N(v)).$$

Since $|T| = t$, the above expression is equivalent to the probability that v is adjacent to t particular vertices. As such:

$$\mathbb{E}[X] = \sum_{v \in V(G)} \left(\frac{d(v)}{n}\right)^t = n^{-t} \sum_{v \in V(G)} d(v)^t.$$

(Here, $d(v)/n$ gives the probability that v is adjacent to some particular vertex, and the power of t

gives the probability that v is adjacent to t particular vertices.) Applying convexity[1] to the above expression yields:

$$\mathbb{E}[X] = n^{-t} \sum_{v \in V(G)} d(v)^t \geq n^{-t} \cdot n^{1-t} \left(\sum_{v \in V(G)} d(v) \right)^t .$$

Bringing the n^{-t} term back into the parentheses:

$$\mathbb{E}[X] \geq n^{1-t} \left(\frac{\sum_{v \in V(G)} d(v)}{n} \right)^t .$$

Note that the parenthetical expression is exactly the average degree d of G. Therefore, we have:

$$\mathbb{E}[X] \geq \frac{d^t}{n^{t-1}}.$$

2. $\binom{n}{r} \left(\frac{m}{n}\right)^t$: We determine the expected number of r-vertex subsets of A with fewer than m common neighbors. Let a set S of r vertices be bad if it has fewer than m common neighbors, and let Y be the number of bad sets $S \subset A$. We claim that

$$\mathbb{E}[Y] > \binom{n}{r} \left(\frac{m}{n}\right)^t .$$

We first consider the probability that any set S of r vertices is a subset of A. Recall from (1) that A is the common neighborhood of T, and so the probability that S is a subset of A is precisely equal to the probability that T is in the common neighborhood of S. In other words:

$$\mathbb{P}(S \subset A) = \left(\frac{\#\text{ of common neighbors of } S}{n} \right)^t .$$

(Here, the parenthetical term gives the probability that all vertices of S are adjacent to a particular vertex, and the exponent t gives the probability that the vertices of S are adjacent to $t = |T|$ particular vertices.) Now suppose that the number of common neighbors of S is less than m (i.e. S is bad). Plugging this into the above expression, we have that the probability of A containing a bad set is strictly less than

$$\left(\frac{m}{n}\right)^t .$$

Note that there are $\binom{n}{r}$ possible sets of size r in G. Therefore, by linearity of expectation, we have that the expected number of bad sets S in A is:

$$\mathbb{E}[Y] < \binom{n}{r} \left(\frac{m}{n}\right)^t .$$

In (1) we determined the expected size of a set A of common neighbors of t vertices of G (i.e. $\mathbb{E}[X]$). In (2) we determined the expected number of subsets of A of size r with fewer than m common neighbors (i.e. $\mathbb{E}[Y]$). We now delete one vertex from each subset of A that has fewer than m common neighbors, and let the resulting set be U. Note that U contains no bad sets; every subset of U of size r has at least m common neighbors. By linearity of expectation, we have that the expected size of U is

$$\mathbb{E}[U] = \mathbb{E}[X] - \mathbb{E}[Y] = \mathbb{E}[X - Y] = \frac{d^t}{n^{t-1}} - \binom{n}{r} \left(\frac{m}{n}\right)^t \geq a.$$

[1] Convexity arguments are often used in combinatorics to simplify expressions that involve sums of powers. The particular theorem that we are invoking here is a corollary to Hölder's Inequality, which states that

$$\sum_{i=1}^{n} |x_i|^p \geq n^{1-p} \left(\sum_{i=1}^{n} |x_i| \right)^p$$

for $p > 1$ and real $x_1, \ldots, x_n$. The interested reader is directed to *The Cauchy-Schwartz Master Class* by J. Michael Steele for all things convexity [6].

Since the expected size of U is at least a, by the same probabilistic arguments we saw in the previous section, G must contain a subset U of at least a vertices. Therefore, we have that G contains a subset U of vertices that satisfy the constraints of the lemma. $\square$

Let us take a moment to review this result. We have shown that we can find some set of U vertices in G (that satisfies the constraints of the lemma) as follows:

1. Look at all common neighborhoods A of sets of t vertices in G.
2. For each set A, identify those subsets of A with fewer than m common neighbors, and delete one vertex from each such subset. Call the resulting set U.
3. We have that the average size of these sets U is

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq a.$$

   Since this is the average size, it must be the case that at least one such set U contains at least a vertices.

With this process and the specific result in mind, let us revisit our university example in greater detail. Rather than the entire student body, this time we will focus in on the math department.

> Suppose we wish the model the relationships between students in the math department of a university. Suppose that there are 60 students in the math department, and each student on average knows 40 other students. We might ask: "What is the largest group of students within which every pair of students has at least 5 mutual friends?" By the lemma of dependent random choice, we have that for some integer $t > 0$:
>
> $$a \leq \frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t = \frac{40^t}{60^{t-1}} - \binom{60}{2}\left(\frac{5}{60}\right)^t.$$
>
> Graphing this expression, we can see that it is maximized for $t = 3$, and plugging this in gives $a \leq 16.75$. Therefore, we know that there exists a group of 16 students wherein every pair of students has at least 5 mutual friends.

This is a powerful result. From only general information about a graph (average degree and number of vertices), we are able to derive some fairly specific information. Furthermore, we are able to achieve this result in constant time. This means that there is no real limitation on the size of graphs to which we may apply dependent random choice[2]. Up until this point, we have only seen contrived applications of this technique. In the following section, we will examine how dependent random choice can be used to solve challenging problems in combinatorics and graph theory.

3 Practical Uses

We begin this section with an examination of the problem of extremal numbers (which we introduced in section 1.2). We then discuss the problem of determining the Ramsey number of the cube, which remains open.

[2]Perhaps an interesting (although maybe not especially meaningful) application of this technique is to the Facebook graph. Since we (or Facebook) know the degree of each vertex (the size of a person's friend group) and the number of users, we can readily determine the average degree of a vertex. From this information, dependent random choice allows us to ascertain the presence and size of mutually connected groups of people. Of course, this technique may fall somewhat short, since the Facebook graph is likely to be quite sparse.

3.1 Turán numbers of bipartite graphs

Recall that the extremal number (also called the Turán number) of a graph H is the maximum number of edges in a graph G on n vertices, such that G does not contain a copy of H. In section 1.2, we summarized results from Mantel and Turán that gave the extremal numbers $\mathrm{ex}(n, K_3)$ and $\mathrm{ex}(n, K_r)$, respectively. We also stated that the extremal number of bipartite graphs with maximum degree r is at most $cn^{2-\frac{1}{r}}$ for some constant c. We shall now prove this result. [The following theorem is found in [2], and the proof incorporates ideas from [2] and [3].]

Theorem 3.1. *Let H be a bipartite graph with parts A and B. If all vertices in B have degree at most r, then $ex(n, H) \leq cn^{2-\frac{1}{r}}$, where c is a constant whose value depends only on H.*

Proof: Let G be a graph with n vertices and at least $cn^{2-\frac{1}{r}}$ edges. We show that (1) G contains a subset of vertices U, where $|U| \geq |A|$ and every subset of r vertices in U has at least $|A| + |B|$ common neighbors, and (2) that this implies that G must contain at least one copy of H.

1. Since $|E(G)| \geq cn^{2-\frac{1}{r}}$ and $d = 2|E(G)|/n$, we have that

   $$d = \frac{2cn^{2-\frac{1}{r}}}{n} = 2cn^{1-\frac{1}{r}}.$$

   We let $t = r$ and $m = |A| + |B|$. Plugging these values into

   $$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t$$

   and simplifying gives:

   $$\frac{\left(2cn^{1-\frac{1}{r}}\right)^r}{n^{r-1}} - \binom{n}{r}\left(\frac{|A| + |B|}{n}\right)^r = (2c)^r - \binom{n}{r}\left(\frac{|A| + |B|}{n}\right)^r.$$

   Recall that $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$. Applying this inequality to the above expression yields:

   $$(2c)^r - \binom{n}{r}\left(\frac{|A| + |B|}{n}\right)^r \geq (2c)^r - \left(\frac{en}{r}\right)^r\left(\frac{|A| + |B|}{n}\right)^r = (2c)^r - \left(\frac{e(|A| + |B|)}{r}\right)^r.$$

   Note that the term

   $$\left(\frac{e(|A| + |B|)}{r}\right)^r$$

   is simply a constant function of H. Therefore, we can always find a sufficiently large value of c such that

   $$(2c)^r - \left(\frac{e(|A| + |B|)}{r}\right)^r \geq |A|.$$

   By the lemma of dependent random choice, the above expression tells us that G must contain a subset of vertices U, where $|U| \geq |A|$ and every subset of r vertices in U has at least $|A| + |B|$ common neighbors.

2. Let $f : A \cup B \to V(G)$ be an embedding of $H = A \cup B$ in G. First, arbitrarily embed the vertices of A in U. (This is possible since in (1), we showed that $|U| \geq |A|$.) Let $B = \{v_1, \ldots, v_{|B|}\}$. We embed the vertices of B sequentially. Suppose v_i is an arbitrary vertex of B. We embed $v_i \in B$ in G. Let $N_i = N(v_i)$ be the vertices of A that are adjacent to v_i. By definition, $|N_i \leq r$. Therefore, $f(N_i)$ is a subset of U of size at most r, and so by (1), $f(N_i)$ has a common neighborhood of size at least $|A| + |B|$. Since we have embedded fewer than $|A| + |B|$ vertices, there must exist some vertex w that is adjacent to all vertices in $f(N_i)$ and is not yet part of our embedding. We let $f(v_i) = w$. Proceeding through all $v_i \in B$ in such a way gives an embedding of H in G. □

We should note that part (2) of the above proof also proves the following embedding lemma:

Lemma 3.2. *Suppose $H = A \cup B$ is a bipartite graph, where the maximum degree of any vertex in B is r. If G is a graph containing a subset U of $|A|$ vertices, where all sets of r vertices in U have at least $|A| + |B|$ common neighbors, then H is a subgraph of G.*

We shall use this lemma in the following section.

3.2 Ramsey number of the cube

Earlier in this paper, as an example of the probabilistic method, we saw a proof of a lower bound on Ramsey numbers $R(k, k)$. We shall now look at a different problem related to Ramsey numbers: the Ramsey number of the (hyper)cube.

Recall that the Ramsey number $R(H)$ is the minimum integer n such that every 2-edge-coloring of K_n (the complete graph on n vertices) contains a monochromatic copy of H. Let Q_r be the graph of the r-dimensional hypercube; in other words, define Q_r to be an r-regular graph on 2^r vertices, where each vertex corresponds to a unique binary string of length r, and two vertices are adjacent if and only if their corresponding binary strings differ by exactly one digit. We prove an upper bound on $R(Q_r)$.

The problem of determining $R(Q_r)$ has been around for well over 30 years. An upper bound of 2^{cr^2} was given by Beck in the early 1980s [7]. Improvements have been made in the subsequent years by a number of mathematicians, and the bound has been chipped down to a polynomial expression in 2^r (i.e. polynomial in the number of vertices of Q_r). We shall use dependent random choice to demonstrate a polynomial upper bound on $R(Q_4)$ of 2^{3r}. The conjectured upper bound on $R(Q_r)$ is linear in 2^r, but this problem remains open. [The following theorem and proof are due to Fox and Sudakov [2].]

Theorem 3.3. $R(Q_r) \leq 2^{3r}$.

Proof: Let $G = K_n$ be a 2-edge-colored graph on $n = 2^{3r}$ vertices. We apply the lemma of dependent random choice to prove that any monochromatic subgraph of G with at least $\frac{1}{2}\binom{n}{2}$ edges must contain a copy of Q_r.

Let G' be the monochromatic subgraph obtained from G by taking all edges of the denser color (i.e. all edges belonging to the most edge-dense color). Note that G' must have at least $\frac{1}{2}\binom{n}{2}$ edges. Furthermore, note that

$$\frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{2^2} \geq \frac{n^2}{2^{7/3}}.$$

Computing the average degree d of G', we have:

$$d \geq \frac{2n^2}{2^{7/3}n} = \frac{n}{2^{4/3}} = 2^{-4/3}n.$$

We let $t = \frac{3}{2}r$ and $m = 2^r$. Plugging these values into

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t$$

and simplifying yields

$$\frac{(2^{-\frac{4}{3}}n)^{\frac{3}{2}r}}{n^{\frac{3}{2}r-1}} - \binom{n}{r}\left(\frac{2^r}{n}\right)^{\frac{3}{2}r} = 2^{-2r} - \binom{n}{r}\left(\frac{2^r}{n}\right)^{\frac{3}{2}r}.$$

Note that $\binom{n}{r} \leq \frac{n^r}{r!}$. Plugging this inequality into the above expression, we have:

$$2^{-2r} - \binom{n}{r}\left(\frac{2^r}{n}\right)^{\frac{3}{2}r} \geq 2^{-2r} - \frac{n^r}{r!}\left(\frac{2^r}{n}\right)^{\frac{3}{2}r} = 2^{-2r}n - \frac{2^{\frac{3}{2}r^2}n^{-\frac{1}{2}r}}{r!}.$$

Finally, plugging in $n = 2^{3r}$ gives:

$$2^{-2r}2^{3r} - \frac{2^{\frac{3}{2}r^2}(2^{3r})^{-\frac{1}{2}r}}{r!} = 2^r - \frac{1}{r!} \geq 2^r - 1 \geq 2^{r-1}.$$

By the lemma of dependent random choice, the above expression tells us that G' must contain a set U of 2^{r-1} vertices, where every subset of U of size r has at least 2^r common neighbors.

Note that the graph Q_r is bipartite. (Let $Q_r = A \cup B$, where vertices in A have associated binary strings with an even number of 1s, and vertices in B have associated binary strings with an odd number of 1s. Since adjacent vertices have binary strings that differ by exactly one digit, vertices with the same parity of 1s cannot be adjacent. Therefore, Q_r is bipartite with parts A and B of size 2^{r-1}.) Also, note that Q_r is r-regular (i.e. every vertex has degree r). By lemma 3.2 from the previous section, it then follows that Q_r is a subgraph of G'. $\square$

Note: Some progress has been made on the related problem of determining the Ramsey number $R(K_s, Q_r)$. In a 2013 paper by Pontiveros et al. [4], it was shown that $R(K_s, Q_r) = (s-1)(2^n - 1) + 1$. While this does not answer the question of whether the upper bound on $R(Q_r)$ is linear in 2^r, it does hint that improvements can likely be made.

4 Concluding Remarks

In this paper, we have introduced the idea of probabilistic methods in graph theory and combinatorics, presented the specific probabilistic tool of dependent random choice, both abstractly and formally, and looked at two applications of this technique (Turán numbers of bipartite graphs and Ramsey numbers of the hypercube). It should come as no surprise that this is just the tip of the iceberg. A paper by Fox and Sudakov [2] (upon which this paper is based) lists a number of other applications of the method of dependent random choice. These include:

- Embedding a 1-subdivision of the complete graph: A 1-subdivision of a graph G is a graph G' resulting from replacing some edges of G with paths of length 3. This problem asks whether any graph with "enough" edges contains a 1-subdivision of the complete graph.

- Ramsey-Turán problem for K_4-free graphs: The Ramsey-Turán number $\mathbf{RT}(n, H, f(n))$ is the maximum number of edges in an n-vertex graph that does not contain the graph H as a subgraph, nor has an independent set of size $f(n)$. This problem addresses the specific question of $\mathbf{RT}(n, K_4, f(n))$, and proves a result for very small $f(n)$.

The applications of dependent random choice and other probabilistic methods to graph-theoretical problems are many and varied. In this paper, we have caught a glimpse of the strength of these techniques. For a more detailed exploration of these ideas, we strongly recommend the survey paper "Dependent Random Choice" by Fox and Sudakov [2].

5 References

[1] Balachandran, Niranjan. "The Probabilistic Method in Combinatorics." Self-published lectures. (2011): 1-93.

[2] Fox, Jacob, and Benny Sudakov. "Dependent random choice." Random Structures & Algorithms 38.12 (2011): 68-99.

[3] Zhao, Yufei. "Graph Theory and Additive Combinatorics." MIT lecture notes. Course 18.217 (Fall 2019).

[4] Fiz Pontiveros, Gonzalo et al. "The Ramsey Number of the Clique and the Hypercube." Journal of the London Mathematical Society 89.3 (2014): 680-702. Crossref. Web.

[5] Zhao, Yufei. "Forbidding a subgraph IV: Dependent random choice." YouTube, May 2020, https://www.youtube.com/watch?v=NpMv0Nqmy3c.

[6] Steele, J. Michael. "The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities." Cambridge University Press, 2004.

[7] J. Beck. “An upper bound for diagonal Ramsey numbers.” Studia Sci. Math. Hungar 18 (1983), 401-406.